

# Abstract Deduction and Inferential Models for Type Theory

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## Abstract

An inferential semantics for full Higher Order Logic (HOL) is proposed. The paper presents a constructive notion of model, that being able to capture relevant computational aspects is particularly suited for the applications of HOL to computer science. The inferential semantics is based on the introduction of new abstract deduction structures (ADS) that express the action of the Comprehension Axiom in a higher order logic proof. The ADSs allow to define an inferential algebra of higher order proof-trees, endowed with two binary operations, the abstraction and the contraction, each consisting of constructive reductions between proofs. Typed formulas are interpreted by trees, and the reductions between proofs correspond to the logical connectives of the interpreted formula. Higher order logic is sound and complete w.r.t. the given inferential semantics.

*Keywords:* Logic in Computer Science, constructive semantics for higher order logic, proof-theory

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## 1. Introduction

In this work an *inferential semantics* for classical Higher Order Logic is proposed which provides a new kind of syntactic models. The formulation of Higher Order Logic considered in this paper is the sequent version of Church's theory of types defined by Miller, Scedrov, Nadathur and Pfenning in the early 1990's (see, e.g., [23]), so that also typed  $\lambda$ -calculus is included in the system.

The main goal of the paper is to introduce a new notion of interpretation that could more easily convey a semantic characterization related to the notion of proof: *inferential semantics* introduces a class of models of Higher Order Logic with such a constructive nature, which is strong enough to allow a completeness result.

The well known papers of Henkin [17] and Andrews [1], [2] provided a semantics for Church's classical type theory by means of *general models*, where the information about the semantical features of the interpreted formula remains mostly implicit. Conversely, *inferential semantics* aims at interpreting a typed formula  $B_\alpha$  through an effectively constructed object, including explicit information, that translates the syntax of  $B_\alpha$ , both the logical part and the typed part, into some precise inferential properties of a sequent tree.

To this aim, a new proof-theoretic analysis of full Higher Order Logic is proposed, based on a formal characterization of the role of the Comprehension-rules ( $\exists$ -R,  $\forall$ -L, i.e. the two rules expressing the Comprehension Axiom) in a sequent tree. Such analysis allows to define an abstraction of the proof-trees resulting in sequences of formulas (called *abstract deduction structures*) which are linked to specific occurrences of Comprehension-rules in the tree. Thus, essentially, *inferential semantics* associates to a formula  $B_\alpha$  a set of *abstract deduction structures* occurring in a sequent tree.

As a consequence of the effective character of the inferential interpretation, a new notion of *meaning* for typed formulas of arbitrary type can be introduced. It is independent of the truth denotation: thus, *inferential semantics* can formally separate truth denotation and meaning. Moreover, this allows the definition of new complexity measures on formulas and proofs, that include both the syntactic and the semantic aspects.

It is worth noting that the role of Higher Order Logic in many application fields, most notably *computer science*, is important and significant: type systems for programming languages and models for the typed  $\lambda$ -calculus are relevant examples, e.g., typed  $\lambda$ -calculus [19] is a part of Church's formulation of Higher Order Logic [8]. Logic programming has also been deeply influenced by Higher Order Logic: the studies of Andrews [1] on higher order theorem provers are at the basis of the works of Miller on  $\lambda$ -Prolog [22]. Thus, the *inferential semantics* proposed

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in this paper could be seen as a contribution to the construction of richer semantical tools that can help reasoning about some computational aspects of various formulations of HOL.

Further relevant developments can be envisaged when we restrict Higher Order Logic to some fragments of a logic programming nature, e.g. the higher order logic programming languages defined in [23], where the fundamental notion of uniform proof is introduced. Bai and Blair [4] and Wolfram [30] gave semantics for the classical higher order Horn clause fragment of  $\lambda$ -prolog. De Marco and Lipton [9] produced a model theory of resolution on Higher Order Hereditary Harrop formulas (HOHH) with uniform proofs, through a constructive algebraic approach. The *inferential semantics* can be used to obtain, in the Higher Order Logic setting, something similar to the s-semantics approach studied by Levi's research group; it was used to have a notion of model meaningful from the computational point of view (see for example [6]). A preliminary study of extending the s-semantics approach to Higher Order Logic was presented in [21].

Moreover, Higher Order Logic is still a relevant topic also in fundamental research in logic: see for example the work of De Marco and Lipton on completeness and cut-elimination in the intuitionistic theory of types [10]. In such a perspective, one should note that the definition of the main tools of *inferential semantics* does not depend on the logic being classical: thus, they should extend reasonably to non-classical logics, and a completeness result for intuitionistic Higher Order Logic w.r.t. *inferential models* is work progress.

### 1.1. Outline of the paper

In Section 2 the syntax and the sequent version  $\mathbf{LK}_\omega$  of higher order logic that have been chosen are presented. In Section 3 the notion of *Comprehension Abstract Deduction Structure* (Comp-ADS) is introduced; this allows to define in Sec. 3.3 the general notion of *inferential algebra* on a fixed kind  $K$  of Comp-ADS's. In Section 4 the *inferential interpretation* of a typed formula in the domains of an inferential algebra is defined. An example of inferential interpretation is given in Appendix A1. In Section 5 the *inferential frames* and the *inferential structures* are introduced: they allow to define a notion of semantical identification between formulas and, provided that the frame is a *sound functional denotation* (Def. 5.3), the notion of truth of a sentence. In Section 6 a particular kind of Comp-ADS is introduced, the *critical chain ADS*. Section 7 and 8 are devoted to the main theorem of the paper, which proves that the inferential algebras based on the critical chain ADS's give a semantics for  $\mathbf{LK}_\omega$ . In Section 9 soundness and completeness of  $\mathbf{LK}_\omega$  w.r.t. inferential semantics are proven. In Section 10 a new notion of *meaning* for formulas of arbitrary type is proposed: it is independent of the truth denotation, and has an effective and declarative character that allows to introduce *semantical complexity measures* on formulas and proofs. In section 11 the work in progress and some forthcoming results are sketched.<sup>1</sup>

## 2. $\mathbf{LK}_\omega$ : a sequent presentation of simple Type Theory

The sequent system  $\mathbf{LK}_\omega$  presented here will be used as a formal setting for expressing full higher order logic. Therefore,  $\mathbf{LK}_\omega$  will be presented in a style a la Church, not as a type assignment theory (as in [18]), and the basic syntax of the *functional type theory*, as defined in [8], will be used.  $\mathbf{LK}_\omega$  will be expressed through classical Gentzen sequent calculus [14] and the chosen Gentzen - style representation will refer to the sequent rules for abstract logic programming languages given in Miller et al. [23] p.130.

For the general notion of higher order proof-theory, we refer to Takeuti(Ch.3 of [27]) and Girard(Ch.'s 3A, 3B of [16]). The choice of the functional version is based on these considerations : i) the efficiency of the  $\lambda$ -formalism in handling typed expressions; ii) the existence of higher order logic programming languages based on functional type theory [23], that can be considered as a possible application field for the semantics presented here; iii) the possibility of some interaction of the presented syntactic-semantic construction within the specific typed lambda calculus setting [19].

### 2.1. Language of $\mathbf{LK}_\omega$ : the structure of types

First, some useful and well established notions on the structure of the types are recalled.

**Definition 2.1.** *Types are:*

- i) *primitive types*:  $o$ , the propositional type and  $i$ , the type for individuals; they will also be called *atomic types*;
- ii) *compound types*: compound types are inductively constructed from primitive types and have the form  $\alpha \rightarrow \beta$ ; according to the use, the type constructor  $\rightarrow$  associates to the right.

◇

<sup>1</sup>Given that, in some sections, completely new notions are introduced, we will put some notes along the text to help the reader in choosing some examples that can be helpful in understanding the definitions.

Then, also referring to [18] p.115 and p.153, every type  $\tau$  can be uniquely expressed in the form :  $\tau_1 \rightarrow \tau_2 \rightarrow \dots \tau_m \rightarrow a$  where, given the right association convention, the redundant parentheses have been omitted. This form nicely represents a condensed structure (or tree) of the type and can be called a *condensed writing* of  $\tau$ .  $a$  is necessarily a primitive type occurrence called the *tail* of  $\tau$ , and the  $\tau_j$ 's are the *premises* of  $\tau$ .<sup>2</sup> The number  $m$  is the *arity* of  $\tau$ . The complexity parameters are the following:

**Definition 2.2. i)** The *order*  $ord(\tau)$  of a type  $\tau$  is defined on the condensed writing of  $\tau$  as follows :  $ord(i) = 0$ ;  $ord(o) = 1$ ; let  $\tau$  be  $\tau_1 \rightarrow \tau_2 \rightarrow \dots \tau_m \rightarrow a$ ,  $a$  atomic type ; then  $ord(\tau) = \max\{ord(\tau_1), \dots, ord(\tau_m)\} + 1$ .

**ii)** The *depth*  $d(\tau)$  of a type  $\tau$  is defined on the condensed writing of  $\tau$  as follows : let a  $\tau$ -*premise sequence* be a sequence of strict inclusions between premises and subpremises of  $\tau$ , starting from  $\tau$ . Then  $d(\tau)$  is the length of the longest  $\tau$ -premise sequence.<sup>3</sup>

**iii)** The *height*  $h(\tau)$  of a type  $\tau$  is defined as follows:  $h(i) = 0$ ;  $h(o) = 1$ ;  $h(\alpha \rightarrow \beta) = \max\{h(\alpha), h(\beta)\} + 1$ .  $\diamond$

The distinction between depth and order is necessary, since individuals and propositions have different orders. Moreover, it is straightforward to prove that  $ord(\tau) \in \{d(\tau), d(\tau) - 1\}$ . Note that for each  $\tau$  with at least one occurrence of  $o$  in it,  $h(\tau) \geq d(\tau)$ . In the literature both the *height* and the *depth* are used in order to measure the complexity of a type, and they express different aspects of such complexity.

## 2.2. Language of $LK_\omega$ : terms and formulas

**Definition 2.3.** The typed  $\lambda$ -terms of  $LK_\omega$  are defined as follows (the type of the term is always indicated by a greek subscript, the index by a latin superscript):

**i)** for each type  $\alpha$  a denumerable set of *free variables of type  $\alpha$* , indexed by  $j \in \mathbb{N}$  :  $b_\alpha^j, \dots$ ;

for each type  $\alpha$  a denumerable set of *bound variables of type  $\alpha$* , indexed by  $j \in \mathbb{N}$  :  $y_\alpha^j, \dots$ ;

for each type  $\alpha$  a denumerable set of *non logical constants of type  $\alpha$* , indexed on  $j \in \mathbb{N}$  :  $f_\alpha^j, \dots$ ;

variables and constants are called *atoms*.  $b_\alpha$  is called *free variable corresponding to the bound variable  $y_\alpha$*  iff they have the same index.

**ii)** *compound terms* are built from atoms using *applications* and  $\lambda$ -*abstractions*:

if  $d_{\alpha \rightarrow \beta}$  and  $m_\alpha$  are terms then  $d_{\alpha \rightarrow \beta} m_\alpha$  is, by application, a term of type  $\beta$ ; applications associate to the left.

If  $x_\alpha$  is a bound variable of the language and  $a_\beta$  an arbitrary term, then  $\lambda_{x_\alpha}[x_\alpha/b_\alpha]a_\beta$  is a term of type  $\alpha \rightarrow \beta$  called the  $\lambda$ -*abstraction* of  $a_\beta$  at  $x_\alpha$ ;  $[x_\alpha/b_\alpha]a_\beta$  indicates the result of the substitution (Definition 2.4 below) of the possible occurrences in  $a_\beta$  of the free variable  $b_\alpha$  corresponding to  $x_\alpha$ . In general it will be briefly written  $\lambda_{x_\alpha} a_\beta$  for  $\lambda_{x_\alpha}[x_\alpha/b_\alpha]a_\beta$ . A *sub-term* of a term  $q_\alpha$  is each term occurring in it. A formula is *closed* if no free variables occur in it. Closed formulas of type  $o$  are also called *sentences*.

$\diamond$

Latin capital letters will be used as a meta-notation for arbitrary typed terms :  $A_\beta, C_\gamma, \dots$ . Moreover, let  $F[...A_\gamma...]$  denote an arbitrary formula having  $A_\gamma$  as sub-formula. For the details of the definition of terms and the well-known intended meaning of  $\lambda$ -expressions, see: [8], [19] Ch1, [3] pp 161-166 and [18] p.140.

Following Church [8] the definition above is used to define both well formed terms and formulas, independently of their type. Informally, an expression  $C_\gamma$  will be called a *term* if considered as a possible sub-term of a more complex expression, instead it will be called a *formula* if considered independently of any context. At the syntactic level no particular status is given to the  $o$ -typed terms  $A_o$ . As specified below, the notion of sequent is necessary to select a specific syntactic property of the formulas of type  $o$ , as the only formulas that may be *isolated formulas* in a sequent.

**Definition 2.4. (substitution)** The expression  $[C_\alpha/M_\alpha]B_\beta$  denotes the result of the uniform replacement of each occurrence of the term  $M_\alpha$  in  $B_\beta$  with  $C_\alpha$ . A term of the form  $(\lambda_{x_\alpha} A_\gamma)M_\alpha$  is called a  $\beta$ -*redex* and its *contractum* is the corresponding term  $[M_\alpha/x_\alpha]A_\gamma$ ; a  $\beta$ -redex is said to  $\beta$ -reduce to its contractum. A term is in  $\lambda$ -*normal form* if

<sup>2</sup>By iterating the definition starting from each premise occurrence  $\tau_j$ , we obtain straightforwardly a family of subpremise and subtail occurrences of  $\tau$ . The occurrences of premises, tail, subpremises, subtails form the set of the *significant components* of  $\tau$ . For example, let  $\tau$  be  $(i \rightarrow o \rightarrow o) \rightarrow (o \rightarrow i) \rightarrow i \rightarrow o$ . If we use capital letters as meta-notation to indicate indexed type occurrences  $\tau$  can be expressed as:  $(I_1 \rightarrow O_2 \rightarrow O_3) \rightarrow (O_4 \rightarrow I_5) \rightarrow I_6 \rightarrow O_7$ . The set of premise and subpremise occurrences is  $\{\tau, I_1 \rightarrow O_2 \rightarrow O_3, O_4 \rightarrow I_5, I_6, I_1, O_2, O_4\}$  and the set of tail and subtail occurrences is  $\{O_7, I_5, O_3\}$ . Concisely, *premises* of  $\tau$  are both the premise and the subpremise occurrences. The *main sequence* in  $\tau$  is the longest  $\tau$ -premise sequence obtained by selecting at each step the leftmost subpremise, and the *main premise* of  $\tau$  is the second element of the main sequence in  $\tau$ .

<sup>3</sup>If compound types are interpreted as varying in sets constructed by power set operation and cartesian product operation starting from the domain of individuals, the depth of  $\tau$  is the longest number of nested power set operations in the nested brackets.

it contains no  $\beta$ -redex; by the *normalization theorems* (see [19]) each term can be  $\beta$ -reduced to a *unique* normal form.  $\diamond$

**Definition 2.5.** (*logical constants*) The following constants, with the associated type, express in the functional formalism the classical logical connectives:

i) *propositional logical constants*:

negation:  $\neg_{o \rightarrow o}$ ; disjunction:  $\bigvee_{o \rightarrow (o \rightarrow o)}$ ; conjunction:  $\bigwedge_{o \rightarrow (o \rightarrow o)}$ ; implication:  $\supset_{o \rightarrow (o \rightarrow o)}$ ; the syntactic symbol "truth"  $\top$  and the syntactic symbol "falseness"  $\perp$ , both of type  $o$ . If  $A_o, B_o$  are arbitrary formulas of type  $o$  the following conventions will be used in the writing:

$\neg A$  for  $\neg_{o \rightarrow o} A_o$ ;  $A \vee B$  for  $(\bigvee_{o \rightarrow (o \rightarrow o)} A_o) B_o$ ;  $A \wedge B$  for  $(\bigwedge_{o \rightarrow (o \rightarrow o)} A_o) B_o$ ;  $A \supset B$  for  $(\supset_{o \rightarrow (o \rightarrow o)} A_o) B_o$ ;  $A \leftrightarrow B$  for  $(A \supset B) \wedge (B \supset A)$ .

ii) *quantifier logical constants*: for each type  $\alpha$  there are an existential constant,  $\exists_{(\alpha \rightarrow o) \rightarrow o}$  and an universal constant  $\forall_{(\alpha \rightarrow o) \rightarrow o}$ . For each arbitrary fixed  $B_o$  the following conventions will be used in the writing:  $\exists x_\alpha B$ , or  $\exists x_\alpha [x_\alpha / t_\alpha] B$  for  $\exists_{(\alpha \rightarrow o) \rightarrow o} \lambda x_\alpha [x_\alpha / t_\alpha] B_o$ ;  $\forall x_\alpha B$ , or  $\forall x_\alpha [x_\alpha / t_\alpha] B$  for  $\forall_{(\alpha \rightarrow o) \rightarrow o} \lambda x_\alpha [x_\alpha / t_\alpha] B_o$ .

$\diamond$

**Definition 2.6.** (*Equality constants*) Following [3] and [20] equality is introduced as a primitive symbol of the language, i.e. for each type  $\alpha$ ,  $=_{\alpha \rightarrow (\alpha \rightarrow o)}$  is given. Usually  $A_\beta =_\beta B_\beta$  stands for  $(=_{\beta \rightarrow (\beta \rightarrow o)} A_\beta) B_\beta$   $\diamond$

**Definition 2.7.** *Atomic formulas* are formulas of type  $o$  whose leftmost symbol is not a logical constant.  $\diamond$

Note that atom and atomic formula are different concepts.  $\top$  and  $\exists_{(\alpha \rightarrow o) \rightarrow o}$  are atoms, but not atomic formulas. An atomic formula may contain formulas of arbitrary complexity among its subterms, an atom has no proper subterm. Moreover, in  $\mathbf{LK}_\omega$  free and bound variables are different syntactic objects. This syntactic distinction will play a technical role in the definition of models; and, following [27], in a *proof-tree* it is also a necessary distinction.

### 2.3. Sequents

A sequent  $S$  is an expression of the form  $\Gamma \vdash \Delta$  where (following [23] and [11])  $\Gamma$  and  $\Delta$  are finite (possibly empty) *sets* of formulas of type  $o$ ;  $\Gamma$  is called the *antecedent* and  $\Delta$  is called the *succedent* of the sequent; if they are non-empty the sequent has the form  $A^1, \dots, A^n \vdash B^1, \dots, B^m$  with  $A^i, B^j$  formulas of type  $o$ . It is well-known that the intended meaning of  $A^1, \dots, A^n \vdash B^1, \dots, B^m$  is the same as the formula  $\bigwedge_i A^i \supset \bigvee_j B^j$ .

**Definition 2.8.** Each occurrence of an  $o$ -typed formula that is delimited by commas, blank space, symbol  $\vdash$ , possible set brackets  $\{, \}$  in a sequent, is called an *isolated formula*.  $\diamond$

In the usual typed language of Church, the distinction between  $o$ -typed formulas and the remaining formulas appears at the metatheoretic level only, where the notion of provability and truth are exclusively applied to formulas of type  $o$ . Through the sequent formalism, a distinction is also introduced at a strictly syntactic level: *only an  $o$ -typed formula may be an isolated formula* in a sequent.

### 2.4. A Sequent Calculus for $\mathbf{LK}_\omega$

For a general discussion of sequent calculus see [16, 27, 28]. The sequent axioms and rules of  $\mathbf{LK}_\omega$  are the following:

(In a sequent  $\Omega, \Delta, \Gamma, \Pi, \Theta, \dots$  will be used as metaexpressions for finite sets of  $o$ -typed formulas and  $A, B, C, D, \dots$  for isolated formulas. The writing  $\Omega, \Delta$  denotes  $\Omega \cup \Delta$ .)

0) *Logical axioms*:

0i)  $A \vdash A$  where  $A$  is an atomic formula;

0ii)  $\vdash \top$   $\perp \vdash$ ;

1) *Logical rules*:

1i) Propositional logical rules:

$$\begin{array}{c}
\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \neg\text{-R} \\
\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge\text{-L} \\
\frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B} \vee\text{-R} \\
\frac{\Gamma \vdash \Delta, A \quad \Theta \vdash \Omega, B}{\Theta, \Gamma \vdash \Delta, \Omega, A \wedge B} \wedge\text{-R} \\
\frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B} \supset\text{-R} \\
\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \neg\text{-L} \\
\frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge\text{-L} \\
\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \vee\text{-R} \\
\frac{A, \Gamma \vdash \Delta \quad B, \Theta \vdash \Omega}{A \vee B, \Theta, \Gamma \vdash \Delta, \Omega} \vee\text{-L} \\
\frac{\Gamma \vdash \Delta, A \quad B, \Theta \vdash \Omega}{A \supset B, \Theta, \Gamma \vdash \Delta, \Omega} \supset\text{-L}
\end{array}$$

1ii) Logical rules for quantifiers:

$$\begin{array}{c}
\frac{[t_\alpha/x_\alpha]A, \Gamma \vdash \Delta}{\forall x_\alpha A, \Gamma \vdash \Delta} \forall\text{-L} \\
\frac{[b_\alpha/x_\alpha]A, \Gamma \vdash \Delta}{\exists x_\alpha A, \Gamma \vdash \Delta} \exists\text{-L} \\
\frac{\Gamma \vdash \Delta, [b_\alpha/x_\alpha]A}{\Gamma \vdash \Delta, \forall x_\alpha A} \forall\text{-R} \\
\frac{\Gamma \vdash \Delta, [t_\alpha/x_\alpha]A}{\Gamma \vdash \Delta, \exists x_\alpha A} \exists\text{-R}
\end{array}$$

where: in  $\forall\text{-L}$ ,  $\exists\text{-R}$ ,  $t_\alpha$  is an arbitrary term and in the corresponding  $\forall x_\alpha A$ ,  $\exists x_\alpha A$ ,  $t_\alpha$  may still occur, that is  $t_\alpha$  may be not fully quantified in  $\forall x_\alpha A$ ,  $\exists x_\alpha A$ ; on the other hand, in  $\forall\text{-R}$ ,  $\exists\text{-L}$ , the free variable  $b_\alpha$  occurring in  $[b_\alpha/x_\alpha]A$  is uniformly replaced in  $\forall x_\alpha A$ ,  $\exists x_\alpha A$  with the bound variable  $x_\alpha$  having the same index, and  $b_\alpha$  does not occur in  $\Gamma, \Delta$ .  $b_\alpha$  is the *proper variable* or *eigenvariable* of the rule.

1iii)  $\lambda$ -rule:

$$\frac{\Gamma' \vdash \Delta'}{\Gamma \vdash \Delta} \lambda$$

where the sets  $\Gamma$  and  $\Gamma'$  and the sets  $\Delta$  and  $\Delta'$  differ only in that zero or more formulas in them are replaced by some formulas to which they are  $\beta$ -reducible. Note that the rule is defined so that the  $\beta$ -reduction may work either upwards or downwards. A term occurrence in the  $\lambda$ -premise, reduced to a  $\beta$ -contractum or replaced with a  $\beta$ -redex by the  $\lambda$ -rule, is called *maximal* if it does not occur as subterm in any different term reduced or replaced by the rule.

2) *Structural rules*:

2i) Weakening rules:

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \text{W-R} \quad \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \text{W-L}$$

2ii) Cut rule:

$$\frac{\Gamma \vdash \Delta, A \quad A, \Theta \vdash \Omega}{\Theta, \Gamma \vdash \Delta, \Omega} \text{Cut}$$

3) *Typed equality axioms*:

3i)  $\vdash t_\alpha =_\alpha t_\alpha$

3ii)  $\vdash (A_o =_o B_o) \leftrightarrow (A_o \leftrightarrow B_o)$

3iii)  $t_\alpha =_\alpha f_\alpha, q_\alpha =_\alpha r_\alpha, f_\alpha =_\alpha r_\alpha \vdash t_\alpha =_\alpha q_\alpha$

3iv)  $t_\alpha =_\alpha f_\alpha \vdash A_{\alpha \rightarrow o} t_\alpha =_o A_{\alpha \rightarrow o} f_\alpha$

3v) *Extensionality axioms*:

$$\forall x_\beta (F_{\beta \rightarrow \alpha} x_\beta =_\alpha G_{\beta \rightarrow \alpha} x_\beta) \vdash (F_{\beta \rightarrow \alpha} =_{\beta \rightarrow \alpha} G_{\beta \rightarrow \alpha})$$

where  $F_{\beta \rightarrow \alpha}, G_{\beta \rightarrow \alpha}$  are arbitrary  $\beta \rightarrow \alpha$ -typed formulas.

4) *Trees*: a *proof-tree*  $P$  in  $\mathbf{LK}_\omega$  is a finite tree of sequents connected by  $\mathbf{LK}_\omega$ -rules having  $\mathbf{LK}_\omega$ -axioms as leaves. In each rule occurrence  $\mathcal{R}$  the upper sequents are called the *premises* of  $\mathcal{R}$ , the lower sequent is called the *conclusion* of  $\mathcal{R}$ . The lowermost sequent  $S$  in  $P$  is called the *root* or *end-sequent* of  $P$ , and  $P$  is an  $\mathbf{LK}_\omega$ -*proof* of  $S$ . A *semiproof-tree*  $Q$  in  $\mathbf{LK}_\omega$  is a tree of sequents connected by  $\mathbf{LK}_\omega$ -rules, that has arbitrary sequents as leaves. A *branch* in a tree  $P$  is a sequence of sequents, mutually connected by a premise-conclusion relation, starting from a  $P$ -leaf and ending at the  $P$ -root; a *semibranch* is a sub-sequence of sequents mutually connected by a premise-conclusion relation in a branch. The *height* of a tree is the greatest number of proof lines in a branch.

**Definition 2.9.** (*Propositions of a rule occurrence*) Let  $\mathcal{R}$  be a rule occurrence in a  $\mathbf{LK}_\omega$ -tree  $P$ , having one of the forms presented above. Then:

i) If  $\mathcal{R}$  is a propositional rule or a logical rule for quantifiers, we call an *auxiliary proposition* of  $\mathcal{R}$  each isolated formula occurrence (Def. 2.8) in a premise of  $\mathcal{R}$ , to which  $\mathcal{R}$  applies the introduced logical symbol, and a *principal proposition* of  $\mathcal{R}$  the resulting isolated formula occurrence in the conclusion of  $\mathcal{R}$ .

ii) If  $\mathcal{R}$  is a  $\lambda$ -rule, we call an *auxiliary proposition* of  $\mathcal{R}$  each isolated formula occurrence in the premise of  $\mathcal{R}$  which includes a sub-formula which is either  $\beta$ -reduced by  $\mathcal{R}$  or replaced with any of its  $\beta$ -redexes by  $\mathcal{R}$ ; we call a *principal proposition* of  $\mathcal{R}$  the corresponding isolated formula occurrence in the conclusion of  $\mathcal{R}$ .  $\diamond$

**Example 2.10.** In the following  $\supset$ -L rule:

$$\frac{\Gamma \vdash \Delta, A \quad B, \Theta \vdash \Omega}{A \supset B, \Theta, \Gamma \vdash \Delta, \Omega} \supset\text{-L}$$

$A, B$  in the premises are the auxiliary propositions,  $A \supset B$  in the conclusion is the principal proposition.

In the following  $\vee$ -R rule:

$$\frac{\Gamma \vdash \Delta, D}{\Gamma \vdash \Delta, D \vee B} \vee\text{-R}$$

$D$  in the premise is the auxiliary proposition, and  $D \vee B$  in the conclusion is the principal proposition;  $B$  in the proposition  $D \vee B$  is not a distinguished proposition, it is the maximal non isolated introduced term of the rule, as it will be defined in Section 6.  $\diamond$

As usual we assume that all the introduced eigenvariables in a proof  $P$  are distinct from one another, and never occur in different roles. Moreover, if the free variable  $b_\alpha$  occurs in the  $\mathbf{LK}_\omega$ -proof  $P$  and is not an eigenvariable, then the uniform replacement of  $b_\alpha$  with an arbitrary term  $t_\alpha$  produces a tree  $P'$  which is an  $\mathbf{LK}_\omega$ -proof.

The presented system can be considered as a proper extension of the system of Miller et al. [23]: this allows to connect the proof-theoretical analysis and the semantical constructions presented in the paper to the theory of higher order abstract logic programming languages presented in [23]. In contrast with [23], the rules for weakening and negation appear in the system, and in the axioms  $A \vdash A$ ,  $\vdash \top$ ,  $\perp \vdash$  the antecedents and the succedents are at most singletons; moreover, in the axiom  $A \vdash A$ , the formula  $A$  must be atomic. The reasons are the following: for a general study of inference in Type Theory it doesn't seem suitable to have only proof trees in which all the information introduced as input in a branch occurs at each level of the branch; conversely, it is desirable to make available proofs in which, at each level, only the information strictly necessary for the inferential step has been introduced as input. That is, even without reaching the formal control of proof resources achieved in Linear Logic [15], it helps to establish a hierarchy in the quality of inference, so that: *proofs in which the information input is mainly given by axioms have a higher inference quality than those in which the information input is mainly given by weakenings*. Furthermore, accepting atomic logical axioms only, makes the information input the minimal to produce the root. These qualitative features will be formalized in the definition of *strong* and *weak propositional inference*, presented in Section 6, and will be connected to the assignment of an *inferential type* to proofs in  $\mathbf{LK}_\omega$ . As it is well known [27, 25], the equality free part of  $\mathbf{LK}_\omega$  admits cut-elimination.

## 2.5. Gödel numbering

A suitable Gödel-numbering for the expressions of the language is fixed, so that to each expression  $E$  is injectively assigned a Gödel-number  $\#E$ . For the general properties of gödelization see, e.g, [26]. Gödelization of syntax is here employed as a standard tool of proof-theory, and has a useful auxiliary role, not a central role. In particular, note that given the expression  $E$ ,  $\#E$  is recursively computable, and a recursive procedure exists which decides if any natural number  $n$  is the code  $\#F$  of an expression  $F$  of the language. Moreover, formulas, sequents, trees, are all included in the gödelization domain.

## 3. Abstract deduction structures (ADS) in $\mathbf{LK}_\omega$ -trees and Inferential Algebras

### 3.1. The centrality of the Comprehension Axiom and Comp-rules

The fundamental goal of inferential semantics will be to model a higher order sentence through suitable instances of the Comprehension Axiom. The Comprehension Axiom is a canonical topic, but a completely new use of it is proposed in this paper. Indeed, the Comprehension Axiom has a very important role: for example the step from the logic of order  $m$  to the logic of order  $m+1$  can be seen as one that adds the possibility of quantification over the

domain constituted by the relations defined by the formulas  $\phi^{(m)}$  of  $m$ -order logic, in which variables of order at most  $m$  occur. A functional writing of the Comprehension Axiom schema is the following ([3]):

$$\forall X_{\gamma 1}^1 \dots \forall X_{\gamma m}^m \exists f_{\alpha 1 \rightarrow \dots \rightarrow \alpha n \rightarrow \beta} \forall Z_{\alpha 1}^1 \dots \forall Z_{\alpha n}^n (f_{\alpha 1 \rightarrow \dots \rightarrow \alpha n \rightarrow \beta} Z_{\alpha 1}^1 \dots Z_{\alpha n}^n =_{\beta} B_{\beta})$$

where  $B_{\beta}$  is an arbitrary  $\beta$ -typed formula, called the *auxiliary formula* of the axiom, having variables included in the set  $\{X_{\gamma 1}^1, \dots, X_{\gamma m}^m, Z_{\alpha 1}^1, \dots, Z_{\alpha n}^n\}$ ; recall that  $=_{\beta}$  has the same meaning as the logical equivalence;  $f_{\alpha 1 \rightarrow \dots \rightarrow \alpha n \rightarrow \beta}$  is the *principal variable* of the axiom and it cannot coincide with any variable in  $B_{\beta}$ .

It is well known that, see e.g. [27] Ch3: in a *sequent version of type theory*, *Comprehension Axioms are fully expressed by  $\exists$ -R,  $\forall$ -L rules, where for any type  $\alpha$  arbitrary formulas  $t_{\alpha}$  can be quantified*. For example:

$$\frac{\frac{\frac{\vdash [c_{\alpha}/x_{\alpha}](B_{\beta} =_{\beta} B_{\beta})}{\vdash \forall x_{\alpha}(B_{\beta} =_{\beta} B_{\beta})} \forall\text{-R}}{\vdash \forall x_{\alpha}((\lambda y_{\alpha}[y_{\alpha}/x_{\alpha}]B_{\beta})x_{\alpha} =_{\beta} B_{\beta})} \lambda}{\vdash \exists f_{\alpha \rightarrow \beta} \forall x_{\alpha}(f_{\alpha \rightarrow \beta} x_{\alpha} =_{\beta} B_{\beta})} \exists\text{-R}$$

where  $c_{\alpha}$  is a free variable and  $B_{\beta}$  an arbitrary formula. The crucial point is the  $\exists$ -R quantification on the formula  $\lambda y_{\alpha}[y_{\alpha}/x_{\alpha}]B_{\beta}$ .

Therefore, the  $\exists$ -R,  $\forall$ -L rules acting on arbitrary formulas of arbitrary types will be called *Comprehension rules* (briefly *Comp-rules*), and the relevant part of higher order inference is concentrated in them. The Comprehension Axiom and its restrictions determine the proof-theoretic strength of higher order systems. It must be noted that the difficulty in giving a syntactic proof of cut elimination for  $\mathbf{LK}_{\omega}$  is due to the occurrence of Comp-rules in the proofs. Moreover, due to Comp-rules, the cut-free part of  $\mathbf{LK}_{\omega}$  does not admit the sub-formula property for any type. For these canonical topics we refer to [27].

### 3.2. Abstract Deduction Structures

To enhance the role of Comp-rules in a  $\mathbf{LK}_{\omega}$ -tree it is possible to define inference paths that abstract from the standard premise-conclusion inference relation between sequents.<sup>4</sup> These new paths can concentrate on the higher order nature of Comp-rule, hiding more specific details of the proofs. In order to express formally this situation, the notion of *abstract deduction structure linked to an  $\mathbf{LK}_{\omega}$ -proof  $P$*  is introduced. That is, new deduction structures are defined that select and exactly represent the inferential contribution of Comp-rules in  $P$ ; moreover, since the transformations performed by Comp-rules in a tree are also transformations of types, a proof can also be seen as a type transforming machine and this is reflected in these new structures. For example, let us consider the following  $\exists$ -R occurrence  $\mathcal{R}$  in a proof  $P$ :

$$\frac{X \vdash Y, E_{\gamma \rightarrow o}(H_{o \rightarrow \gamma}((F_{(\alpha \rightarrow \beta) \rightarrow o} G_{\alpha \rightarrow \beta} \wedge y_{(\nu \rightarrow \mu) \rightarrow o} b_{\nu \rightarrow \mu}) \supset \forall x_{\gamma}(B_{\gamma \rightarrow o} x_{\gamma} \vee D_o))) \wedge C_{\alpha \rightarrow o} N_{\alpha}}{X \vdash Y, \exists w_{\gamma}(E_{\gamma \rightarrow o} w_{\gamma} \wedge C_{\alpha \rightarrow o} N_{\alpha})}$$

where the existential witness for  $w_{\gamma}$  is the term  $t_{\gamma} \equiv$

$$(H_{o \rightarrow \gamma}((F_{(\alpha \rightarrow \beta) \rightarrow o} G_{\alpha \rightarrow \beta} \wedge y_{(\nu \rightarrow \mu) \rightarrow o} b_{\nu \rightarrow \mu}) \supset \forall x_{\gamma}(B_{\gamma \rightarrow o} x_{\gamma} \vee D_o))).$$

It is evident that a large set of formulas and types occurring in  $t_{\gamma}$ , which are introduced by many different rules above the considered  $\exists$ -R, are deleted and collapsed into the *quantified atom*  $w_{\gamma}$ . In particular, it is worth noting the action on the types occurring in the proof: there may be an arbitrarily large set of types occurring above  $\mathcal{R}$  and in  $t_{\gamma}$ , that do not occur below  $\mathcal{R}$ , due to the action of  $\mathcal{R}$ . No logical rule different from a Comp-rule may have a similar transformation power: propositional rules preserve the intrinsic structure of each term on which they act, and  $\forall$ -R,  $\exists$ -L quantifier rules only produce a renaming of their eigenvariables. Therefore, in a cut-free proof  $P$  in  $\mathbf{LK}_{\omega}$ , *Comp-rules freely transform the sets of types of subterms occurring in their premise*, i.e. they send a term of type  $\alpha$  occurring in the premise into a term of type  $\beta$  different from  $\alpha$  in the conclusion, and in general no constraint exists that allows to determine  $\alpha$  starting from the  $\beta$ -typed term in the conclusion. Such type transformation is possible through the  $\lambda$ -rule too, but in this case only formulas including  $\beta$ -redexes or  $\beta$ -contractums are involved. As to a comparison with the deletion power of cut-rule, it can be observed that this rule has several constraints: two occurrences of the deleted formula must exist, which may be only isolated  $o$ -typed

<sup>4</sup>Definition 3.1 of *Comprehension Abstract Deduction Structure linked to a proof  $P$*  is the first key definition of the paper; after a first reading of Def. 3.1, it could be useful to see directly Example 3.4. Note that the formalizations between definition 3.1 and example 3.4 aim to guarantee a notion of *kind* which is not ambiguous, even if the intended meaning of such concept is sufficiently intuitive.

formulas, each in a different sequent, and located antisymmetrically. Differently, there is no constraint to the action of a Comp-rule. Moreover, we must remember that cut-rules can be almost completely eliminated, with the exception of atomic cuts, *still preserving the same deduction power*; while, it is well known that the elimination of Comp-rules would *completely destroy the relevant higher order deduction power*.

**Definition 3.1. i)** Let  $P$  be a fixed  $\mathbf{LK}_\omega$ -proof tree. A *Comprehension Abstract Deduction Structure linked to  $P$*  (Comp-ADS in  $P$ ) is any finite sequence  $\mathcal{D}$  whose elements are ordered pairs of formula occurrences in  $P$ , built as follows:

- i.1) a set  $\mathbf{R}$  of Comp-rule occurrences  $\{\mathcal{R}_1, \dots, \mathcal{R}_h\}$ ,  $h \geq 1$ , has been selected in a single branch of  $P$ , such that  $\mathcal{R}_c$  occurs above  $\mathcal{R}_{c+1}$  in the branch;
- i.2) having fixed  $\mathbf{R}$ , a sequence  $\mathcal{D}$  of ordered pairs of formula occurrences  $\langle (t_1, t'_1), \dots, (t_h, t'_h) \rangle$  is chosen, such that  $t_j$  occurs as subterm in the auxiliary proposition of  $\mathcal{R}_j$  and  $t'_j$  occurs as subterm in the principal proposition of  $\mathcal{R}_j$ ;

ii) We define the *Comp-measure* of  $\mathcal{D}$  to be the cardinality  $h$  of the associated set of Comp rule occurrences  $\mathbf{R}$ .  $t_1$  in  $\mathcal{D} \equiv \langle (t_1, t'_1), \dots, (t_h, t'_h) \rangle$  is the  *$\mathcal{D}$ -axiom*,  $t_h$  is the  *$\mathcal{D}$ -theorem*. If  $h=1$  the  *$\mathcal{D}$ -axiom* and the  *$\mathcal{D}$ -theorem* coincide. Any formula occurrence in an element of  $\mathcal{D}$  is also called a  *$\mathcal{D}$ -formula*.  $\diamond$

*$\mathcal{D}$ -formulas* can be seen as ordered by the places they take in the sequence obtained from  $\mathcal{D}$  by deleting the parentheses of the pairs. Thus, for the sake of brevity, we can sometimes mention the *first  $\mathcal{D}$ -formula*, the *last  $\mathcal{D}$ -formula* and so on.

Comp ADS's emphasize the role of the Comp-rules in a deduction. However, no formula in an ADS element, including its so called "theorem" needs actually be a theorem in the theory of types.

**Remark 3.1.1** The following properties immediately hold for any Comp-ADS  $\mathcal{T} \equiv \langle (A_{\alpha_1}^1, B_{\beta_1}^1), \dots, (A_{\alpha_r}^r, B_{\beta_r}^r) \rangle$  in a tree  $P$ :

- i) The elements of  $\mathcal{T}$  occur in the same branch of  $P$ , and  $\mathcal{T}$  is ordered with respect to  $P$ , i.e. if  $C_{\gamma_s}^s \in \{A_{\alpha_s}^s, B_{\beta_s}^s\}$  and  $m > n$  then  $C_{\gamma_m}^m$  occurs in  $P$  below  $C_{\gamma_n}^n$ .
  - ii) Both the  $\mathcal{T}$ -axiom  $A_{\alpha_1}^1$  and the  $\mathcal{T}$ -theorem  $A_{\alpha_r}^r$  occur in a Comp-rule auxiliary proposition.
- $\diamond$

Not all the Comp ADS's in a given proof  $P$  are equivalently relevant. We are interested in those ADS-properties including significant information and that may hold in a large class of proofs. Moreover, those properties that can select a relevant subset of Comp ADS's in  $P$ , should be decidable. Let us formalize the notion of Comp ADS through the language of arithmetical systems that are able to represent the recursive functions (see [7]). First, we observe that, for each arbitrary  $\mathbf{LK}_\omega$ -proof, the property "*the sequence  $\mathcal{M}$  is a Comp-ADS in  $P$* " is a primitive recursive property. Indeed: " $\{\mathcal{R}_1, \dots, \mathcal{R}_h\}$  is a set of Comp-rule occurrences in a single branch of  $P$ " (point i.1) of Def 3.1) can be expressed by a  $\Delta_0$ -formula  $B_1(\#\mathcal{R}_1, \dots, \mathcal{R}_h, \#P)$  which is a boolean combination of recursive primitive predicates applied to arithmetical terms; analogously, the constraints that the sequence  $\mathcal{D} \equiv \langle (t_1, t'_1), \dots, (t_h, t'_h) \rangle$  must respect inside  $\mathbf{R} \equiv \{\mathcal{R}_1, \dots, \mathcal{R}_h\}$  (point i.2) of Def 3.1) can be described by a  $\Delta_0$ -formula  $B_2(\#\langle (t_1, t'_1), \dots, (t_h, t'_h) \rangle, \#\mathbf{R})$ . Note that, given  $\mathbf{R}$ , and only assuming Def. 3.1,  $\mathcal{D}$  is not determined: in principle, infinitely many procedures can be defined, that select from  $\mathbf{R}$  different sequences  $\mathcal{D}$ 's respecting the prescribed constraints; analogously, given  $P$ , and only assuming Def. 3.1,  $\mathbf{R}$  is not determined. Otherwise, the notion of Comp-ADS would be trivial. We wish to formalize the choice process that has produced  $\mathbf{R}$  from  $P$  and then  $\mathcal{D}$  from  $\mathbf{R}$ , with the following provisos: we are looking for the definition of a choice rule across *all* proofs, and for an *effective* choice rule, i.e. that can be implemented by a Turing machine.

**Definition 3.2.** Let  $P$  be any arbitrary  $\mathbf{LK}_\omega$ -proof. Recalling Definition 3.1, if  $\mathbf{R} \equiv \{\mathcal{R}_1, \dots, \mathcal{R}_h\}$  and  $\mathcal{D} \equiv \langle (t_1, t'_1), \dots, (t_h, t'_h) \rangle$ , let the following  $\Delta_0$ -formulas  $B_1(\#P, \#\{\mathcal{R}_1, \dots, \mathcal{R}_h\})$  and  $B_2(\#\mathbf{R}, \#\langle (t_1, t'_1), \dots, (t_h, t'_h) \rangle)$  be constructed in the language of Robinson's Arithmetic  $\mathbf{Q}$ , such that:  $B_1$  means " $\{\mathcal{R}_1, \dots, \mathcal{R}_h\}$  is a set of Comp-rule occurrences in a single branch of  $P$  that respects point i1) of Def 3.1",  $B_2$  means " $\mathcal{D} \equiv \langle (t_1, t'_1), \dots, (t_h, t'_h) \rangle$  is a sequence of pairs of terms that is extracted from  $\mathbf{R}$  respecting point i2) of Def. 3.1". Let  $G(\#P, \#\mathbf{R})$  be any  $\Delta_0$ -formula describing some properties of  $\mathbf{R}$  w.r.t.  $P$  that are not already described by  $B_1$  and let  $E(\#P, \#\mathbf{R}, \#\mathcal{D})$  be any  $\Delta_0$ -formula describing some properties of  $\mathcal{D}$  w.r.t.  $\mathbf{R}$  and  $P$  that are not already described by  $B_2$ . Then a *Comp ADS choice criterion in  $P$*  is a  $\mathbf{Q}$ -formula  $F(\#P, x)$ ,  $x$  free variable, of the form:

$$\exists r [B_1(\#P, r) \wedge B_2(r, x) \wedge G(\#P, r) \wedge E(\#P, r, x)]$$

We say that  $\mathcal{T}$  is a *Comp ADS* in  $P$  chosen by the criterion  $F$  if the following conditions hold:

j)  $\mathbf{Q} \vdash F(\#P, \# \mathcal{T})$ .

jj) Let  $\Phi(\#P, z, y) \equiv B_1(\#P, z) \wedge B_2(z, y) \wedge G(\#P, z) \wedge E(\#P, z, y)$  and let  $\exists y \Phi(\#P, \#R, y)$  be  $\mathbf{Q}$ -provable. Then an effective procedure  $\pi$  is  $\mathbf{Q}$ -derivable from  $\exists y \Phi(\#P, \#R, y)$  that, having  $P$  and  $R$  as input, gives as output a finite set  $\{\mathcal{D}_k\}$  of sequences such that  $\Phi(\#P, \#R, \# \mathcal{D}_k)$  is  $\mathbf{Q}$ -provable.  $\diamond$

The point jj) takes into account that, as to the effective choice process, assuming  $P$  as fixed, the relevant input is  $R$  and the relevant output is a finite set  $\{\mathcal{D}_k\}$  of sequences. Observe that, given  $P$ , many *Comp ADS*'s may be selected in  $P$  through a same choice criterion. For example, we may have  $\mathbf{Q} \vdash \Phi(\#P, \#R_1, \# \mathcal{D}_1)$  and  $\mathbf{Q} \vdash \Phi(\#P, \#R_2, \# \mathcal{D}_2)$  for two different *Comp-rule* sets  $R_1$  and  $R_2$  in  $P$  and two different *Comp-ADS*'s  $\mathcal{D}_1, \mathcal{D}_2$  extracted resp. from  $R_1$  and  $R_2$ , selected by a same choice criterion  $F$ ; that is, formally, both  $\mathbf{Q} \vdash F(\#P, \# \mathcal{D}_1)$  and  $\mathbf{Q} \vdash F(\#P, \# \mathcal{D}_2)$  hold. Also the case  $\mathbf{Q} \vdash \Phi(\#P, \#R_1, \# \mathcal{D}_3)$  and  $\mathbf{Q} \vdash \Phi(\#P, \#R_1, \# \mathcal{D}_4)$  is possible, where two different *Comp-ADS*'s  $\mathcal{D}_3, \mathcal{D}_4$ , are extracted from  $R_1$  by  $F$ . On the other hand, the conditions expressed by any choice criterion  $H(\#P, x)$  may be so narrow that *no Comp-ADS  $\mathcal{H}$  exists in  $P$  such that  $\mathbf{Q} \vdash H(\#P, \# \mathcal{H})$* , even if many *Comp-rule* sequences respecting point 3.1.i.1 of Def.3.1 occur in  $P$ . Moreover, different sentences with the form of a choice criterion may produce the same choices. We formalize this last fact through the notion of *kind*.

**Definition 3.3.** A *kind*  $K$  of *Comp ADS* is a class of choice criteria that are  $\mathbf{Q}$ -provably equivalent. We briefly say that  $\mathcal{T}$  is a *Comp ADS* in a proof  $P$  which belongs to the kind  $K$  if  $\mathbf{Q} \vdash F(\#P, \# \mathcal{T})$  for any choice criterion  $F(\#P, x)$  which belongs to  $K$ . A kind  $K_1$  is stronger than the kind  $K_2$  if any sentence of  $K_1$   $\mathbf{Q}$ -implies any sentence of  $K_2$ , and the converse does not hold. A *Comp-ADS*  $\mathcal{T}$  is maximal in  $P$  if it is not included as a sub-sequence in a different *Comp-ADS* of the same kind.  $\diamond$

**Example 3.4.** Consider the following proof  $P$ :

$$\frac{\frac{\frac{z_o \vdash z_o}{z_o \vdash \exists x_o(x_o)} \exists\text{-R} \quad A_{o \rightarrow o} y_o \vdash A_{o \rightarrow o} y_o}{A_{o \rightarrow o} y_o, z_o \vdash \exists x_o(x_o) \wedge A_{o \rightarrow o} y_o} \wedge\text{-R}}{A_{o \rightarrow o} y_o, z_o \vdash \exists_{(o \rightarrow o) \rightarrow o} \lambda u_o (\exists x_o(x_o) \wedge \mathbf{u}_o)} \exists\text{-R}}{\frac{A_{o \rightarrow o} y_o, z_o \vdash \exists w_{(o \rightarrow o) \rightarrow o} (w_{(o \rightarrow o) \rightarrow o} \lambda u_o (\exists x_o(x_o) \wedge \mathbf{u}_o))}{\neg \exists w_{(o \rightarrow o) \rightarrow o} (w_{(o \rightarrow o) \rightarrow o} \lambda u_o (\exists x_o(x_o) \wedge \mathbf{u}_o)), A_{o \rightarrow o} y_o, z_o \vdash} \neg\text{-L}}{\forall h_o(\mathbf{h}_o), A_{o \rightarrow o} y_o, z_o \vdash} \forall\text{-L}} \exists\text{-R}$$

Where at the fourth line from the top we have explicitly written  $\exists_{(o \rightarrow o) \rightarrow o} \lambda u_o (\exists x_o(x_o) \wedge \mathbf{u}_o)$ , without using the abbreviation  $\exists u_o (\exists x_o(x_o) \wedge \mathbf{u}_o)$ . The boldface term occurrences constitute the following *Comp-ADS*  $\mathcal{M}$  in  $P$ .  $\mathcal{M}$  is the sequence  $\langle (A_{\alpha_r}^r, B_{\beta_r}^r) \rangle \equiv \langle (A_{o \rightarrow o}, u_o), (u_o, u_o), (u_o, h_o) \rangle$ , having the following (informally stated) choice criterion: “Choose as  $R$  any *Comp rule* set in  $P$  that respects Def. 3.1. Then each  $\mathcal{M}$ -formula is the element of a given term ancestor-descendant relation in  $P$ ” (the ancestor-descendant relation between terms in a  $\mathbf{LK}_0$ -proof will be extensively defined in Section 6, but its essential meaning is intuitive).  $F$  defines the *ADS*'s kind  $K_1$ . Moreover,  $\mathcal{M}$  has *Comp-measure* 3, it is maximal, the  $\mathcal{M}$ -axiom is  $A_{o \rightarrow o}$  and the  $\mathcal{M}$ -theorem is the occurrence of  $u_o$  in the last pair. Note that types have been transformed alongside  $\mathcal{M}$  from the  $\mathcal{M}$ -axiom type  $o \rightarrow o$  to the  $\mathcal{M}$ -theorem type  $o$ . In  $P$  various kinds of *Comp-ADS*'s are possible. Indeed, consider this (informally stated) criterion  $G$ : “Choose as  $R$  any *Comp rule* set in  $P$  that respects Def. 3.1. Then each *ADS-formula* is the element of a given term ancestor-descendant relation in  $P$ . Moreover, any *Comp-principal proposition* in which any *ADS-formula* different from the last one occurs, must be the auxiliary proposition of a propositional logical rule in  $P$ ”. Such criterion gives a kind  $K_2$  which is strictly included in  $K_1$ , thus  $K_2$  is stronger than  $K_1$  and  $\mathcal{M}$  does not belong to  $K_2$ . In fact,  $u_o$  in the first pair of  $\mathcal{M}$  does not respect the  $K_2$ -criterion. The following sub-sequence  $\mathcal{H}$  of  $\mathcal{M}$ :  $\mathcal{H} \equiv \langle (u_o, u_o), (u_o, h_o) \rangle$  belongs to  $K_2$ . Since  $K_2$  is included in  $K_1$ ,  $\mathcal{H}$  also belongs to  $K_1$ .  $\diamond$

Many different kinds of *Comp ADS*'s are possible in a proof  $P$ . However, also their relevance can be very different. In particular, to obtain an interesting notion of inferential models, a much finer kind of *Comp-ADS*'s is needed than that ones shown in Example 3.4. In order to select *Comp-ADS*'s including relevant information a proof-theoretical analysis is required and it will be presented in Section 6.

Through the *Comp-ADS*'s an association can be established between the deduction performed in an  $\mathbf{LK}_0$ -tree  $P$  and types:

**Definition 3.5.** (*Inferential type of a Comp-ADS in P*)

- i) Let  $\mathcal{T}$  be a Comp-ADS in a  $\mathbf{LK}_\omega$ -proof  $P$ , with  $(P, \mathcal{T})$ -axiom  $A_\gamma$  and  $(P, \mathcal{T})$ -theorem  $B_\delta$ . The *inferential type*  $\tau(\mathcal{T})$  of  $\mathcal{T}$  is the type  $\gamma \rightarrow \delta$ . If  $\mathcal{T}$  has Comp-measure 1, the inferential type is that of the  $\mathcal{T}$ -axiom.
- ii) Fixing a kind  $K$  of Comp-ADS in  $P$ , a Comp-ADS  $\mathcal{T}$  is a *main Comp-ADS* in  $P$  if it has an inferential type  $\tau(\mathcal{T})$  of the highest height  $h(\tau)$  in the kind.  $P$  is *monic with respect to  $K$*  if it has a unique main Comp-ADS of kind  $K$ . If  $P$  is monic w.r.t.  $K$  with main Comp-ADS  $\mathcal{T}$ , then  $\tau(\mathcal{T})$  is also the *inferential type of  $P$  with respect to  $K$* . In a  $K$ -monic tree  $P$  a Comp-rule occurrence is called *main* if it includes, in the auxiliary proposition, the theorem of the main  $K$ -Comp-ADS of  $P$ .

◇

The notion of inferential type is related to the deduction expressed by the ADS: it types the link between the ADS-axiom and the ADS-theorem. Moreover, it relates types and proof-trees in a new way.

### 3.3. Inferential Algebras based on Comprehension-Abstract Deduction Structures

We wish to construct models which manipulate sequent-trees and that are linked to the notion of Comp-ADS.<sup>5</sup> Thus, we need particular objects called *potential proof-trees* (not to be confused with real proof-trees).

**Definition 3.6.** For each type  $\alpha$  new list of formal metavariables  $\dots Y_\alpha^j \dots, j \in \mathbf{N}$ , is added to the language of  $\mathbf{LK}_\omega$ , with the stipulation that they never occur in the  $\mathbf{LK}_\omega$ -proofs. They are called *syntactic parameters*. ◇

**Definition 3.7.** A *potential proof-tree*  $P$  in  $\mathbf{LK}_\omega$  is a tree in which some leaves are *potential logical axioms* of the form  $\Gamma \vdash B$ , where  $B$  is an arbitrary  $o$ -typed formula and  $\Gamma$  a syntactic parameter of type  $o$  which is potentially replaceable by any set of formulas of type  $o$ . In a potential proof-tree syntactic parameters can be employed as free variables. ◇

As to the construction of the potential proof-trees of the inferential domains (Definition 3.9), only syntactical parameters will be employed as free variables. This because in the assignments  $\varphi$  of the free variables of the  $\mathbf{LK}_\omega$ -language, it shouldn't be the case that for a free variable  $a_\beta$ ,  $a_\beta$  itself occurs in  $\varphi(a_\beta) \in \mathbf{D}_\beta$ , where  $\mathbf{D}_\beta$  is a domain composed by potential proof-trees.

Having defined the Comp-ADS's and their inferential types, an informal sketch of the aims of inferential models can be stated: formulas of full higher order logic are modeled through instances of the Comprehension Axiom, by employing domains formed by  $\mathbf{LK}_\omega$ -potential proof trees, in which suitable kinds of Comp-ADS's occur. Moreover, a constructive inferential algebra (Definition 3.9) between the trees of the domains is defined, so that logical connectives are modeled by corresponding operations between trees.

**Definition 3.8.** Let  $A_\delta$  be a formula of the form  $B_{\gamma \rightarrow \delta} C_\gamma$  where  $C_\gamma$  is an atom. Then we say that  $B_{\gamma \rightarrow \delta}$  is the *significant leftmost component* of  $A_\delta$ . ◇

**Definition 3.9.** Let  $K$  be a kind of Comp-ADS such that, if  $F$  is a choice criterion (Def. 3.2) in  $K$ , for each type  $\beta$  a denumerable set of pairs  $(P, \mathcal{T})$  exists with  $\mathbf{Q} \vdash F(\#P, \#\mathcal{T})$  and *inferential type*  $(\mathcal{T}) \equiv \beta$ ,  $\mathbf{Q}$  Robinson's Arithmetic. Then, the class of  *$K$ -based inferential domains*  $\{\mathbf{K}\mathbf{D}_\alpha\}_{\alpha \in \text{types}}$  of modular  $\mathbf{LK}_\omega$ -trees, and an *inferential algebra* on  $\cup_{\alpha \in \text{types}} \mathbf{K}\mathbf{D}_\alpha$ , are defined as follows:

- i) a recursive bijection:  $g_{(\alpha, \beta)} : \{A_\alpha : A_\alpha \text{ closed formula}\} \times \{A_\beta : A_\beta \text{ closed formula}\} \rightarrow \{A_{\alpha \rightarrow \beta} : A_{\alpha \rightarrow \beta} \text{ closed formula}\}$  is given for each pair  $(\alpha, \beta)$  of types, which is the *canonical bijection* of the algebra. The formulas in which syntactic parameters occur, are also included in the domain and in the codomain of  $g$ .
- ii)  $\mathbf{K}\mathbf{D}_o$  (resp.  $\mathbf{K}\mathbf{D}_i$ ) is a denumerable set of  $\mathbf{LK}_\omega$ -potential proof trees (Definition 3.7) that have a main Comp-ADS of kind  $K$ , that are  $K$ -monic and have  $K$ -inferential type  $o$  (resp.  $i$ ) (Definition 3.5); this is the set of  *$K$ -modular trees of inferential type  $o$*  (resp.  $i$ ).
- iii) Let  $\mathbf{K}\mathbf{D}_\gamma$  be the set of  $K$ -modular trees of inferential type  $\gamma$  and let  $\mathbf{K}\mathbf{D}_\delta$  be the set of  $K$ -modular trees of inferential type  $\delta$ ; then a recursive procedure  $\Rightarrow$  is given, acting on the pairs  $\langle Q_\gamma, Q_\delta \rangle \in \mathbf{K}\mathbf{D}_\gamma \times \mathbf{K}\mathbf{D}_\delta$ , such that:
  - iii.1)  $\Rightarrow \langle Q_\gamma, Q_\delta \rangle$  is a tree of  $K$ -inferential type  $\gamma \rightarrow \delta$ .
  - iii.2) The theorem  $B_\delta$  of the main Comp-ADS of  $\Rightarrow \langle Q_\gamma, Q_\delta \rangle$  has the form  $B_\delta \equiv g(F_\gamma, E_\delta)C_\gamma$ , where  $C_\gamma$  is an atom and  $E_{\gamma \rightarrow \delta} \equiv g(F_\gamma, E_\delta)$  is the *canonical component* of  $B_\delta$ , such that:  $F_\gamma$  (resp.  $E_\delta$ ) is the significant leftmost component (Def. 3.8) of the theorem of the main Comp-ADS of  $Q_\gamma$  (resp.  $Q_\delta$ ) if  $\gamma$  (resp.  $\delta$ ) is not primitive;  $F_\gamma$

<sup>5</sup>Definition 3.9 of *inferential algebra based on a kind  $K$  of Comp-ADS* is the second key definition of the paper. To visualize operations between trees may be difficult. For a synthetic insight, it could be useful to go directly to Section 7 and see (omitting the details) the constructions in STEP 1 of Lemma 7.2, or to go to the Appendix and see the first part of Example A1. Also the comments below Remark 3.3 can help to imagine how in any inferential algebra the trees of higher inferential type are produced from trees of lower inferential types.

(resp.  $E_\delta$ ) is the theorem of the main Comp-ADS of  $Q_\gamma$  (resp.  $Q_\delta$ ) if  $\gamma$  (resp.  $\delta$ ) is primitive. In the trees of the domains  ${}^K\mathbf{D}_o$  and  ${}^K\mathbf{D}_i$  the *canonical component* of the main Comp-ADS theorem coincides with the theorem itself.

iv) The set of the trees of the form  $\Rightarrow \langle Q_\gamma, Q_\delta \rangle$  constitutes the domain  ${}^K\mathbf{D}_{\gamma \rightarrow \delta}$ . The procedure  $\Rightarrow$  defines a recursive function  ${}^K\mathbf{D}_\gamma \times {}^K\mathbf{D}_\delta \rightarrow {}^K\mathbf{D}_{\gamma \rightarrow \delta}$ .  $Q_\gamma \Rightarrow Q_\delta$  denotes the tree  $\Rightarrow \langle Q_\gamma, Q_\delta \rangle$ , and  $\Rightarrow$  is called the *abstraction operation* of the inferential algebra.  $Q_\gamma, Q_\delta$  are the *abstracted trees* of the *abstraction tree*  $Q_\gamma \Rightarrow Q_\delta$ .

v) Having constructed the domains  $\{{}^K\mathbf{D}_\alpha\}_{\alpha \in \text{types}}$ , for each pair of types of the form  $\gamma$  and  $\gamma \rightarrow \delta$  a recursive function  $*$ :  ${}^K\mathbf{D}_{\gamma \rightarrow \delta} \times {}^K\mathbf{D}_\gamma \rightarrow {}^K\mathbf{D}_\delta$  is given, such that the tree  $*\langle Q_{\gamma \rightarrow \delta}, Q_\gamma \rangle$  is a tree of  $K$ -inferential type  $\delta$ .  $Q_{\gamma \rightarrow \delta} * Q_\gamma$  denotes the tree  $*\langle Q_{\gamma \rightarrow \delta}, Q_\gamma \rangle$  and  $*$  is called the *application operation* of the inferential algebra.  $Q_\gamma, Q_{\gamma \rightarrow \delta}$  are the *contracted trees* in the *application tree*  $Q_{\gamma \rightarrow \delta} * Q_\gamma$ .

vi) The set  $\cup_{\alpha \in \text{types}} {}^K\mathbf{D}_\alpha$  of  $K$ -modular trees of arbitrary inferential types with the operations abstraction  $\Rightarrow$  and application  $*$ , is called an *inferential algebra of  $\mathbf{LK}_\omega$ -potential proof trees based on the Comp-ADS's of kind  $K$* , which we also write  $\langle g, \{{}^K\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$ .

◇

### Remark 3.9.1

i) Each tree  $Q_{\gamma \rightarrow \delta} \in {}^K\mathbf{D}_{\gamma \rightarrow \delta}$  defines a recursive function  $Q_{\gamma \rightarrow \delta} * (\cdot) : {}^K\mathbf{D}_\gamma \rightarrow {}^K\mathbf{D}_\delta$  such that  $Q_{\gamma \rightarrow \delta} * (\cdot)$  applied to the argument  $Q_\gamma$  is  $Q_{\gamma \rightarrow \delta} * Q_\gamma$ .

ii) The only notion of identity which a priori holds between the trees of the algebra is the syntactical identity.

iii) At point iii) of def. 3.9 the difference between primitive and compound types as to the inferential types of the trees in the argument of  $\Rightarrow$  depends on the fact that the trees of the domains  ${}^K\mathbf{D}_i$  and  ${}^K\mathbf{D}_o$  are not obtained by abstraction operation, and their main Comp-ADS theorem has a fixed type but may have an arbitrary form.

◇

The abstraction operation  $\Rightarrow$  of the algebra acts at two levels: at the level of tree-structure, by producing in the abstraction tree  $Q_\gamma \Rightarrow Q_\delta$  a main Comp-Ads of kind  $K$  and inferential type  $\gamma \rightarrow \delta$ , and at the level of the formula-structure, by producing in the abstraction tree  $Q_\gamma \Rightarrow Q_\delta$  a main Comp-ADS theorem  $B_\delta$  including a subformula of type  $\gamma \rightarrow \delta$  in a canonical way. It is the canonical bijection  $g$  that describes the behaviour of the algebra at the formula level. Observe that the main Comp-Ads theorem of a monic tree (briefly: the *ADS-theorem* of the tree) with inferential type  $\gamma \rightarrow \delta$  cannot have type  $\gamma \rightarrow \delta$ . Only in the case of primitive types a tree of inferential type  $i$  or  $o$  has the *ADS-theorem* with type  $i$  or  $o$ . Thus, in order to obtain in  $Q_\gamma \Rightarrow Q_\delta$  an *ADS-theorem* that expresses a kind of abstraction operation between the *ADS-theorems* of the two abstracted tree, we select the leftmost component. It is worth noting that, this way, some choices for the canonical bijection  $g$  are possible which would produce algebras where in the abstracted trees  $Q_\gamma \Rightarrow Q_\delta$  the *ADS-theorem* includes a  $\lambda$ -abstraction involving the (leftmost components of the) *ADS-theorems* of the abstracted trees (see Section 11.2). Thus, given  $Q_\gamma, Q_\delta$ , on one hand the inferential type of  $\Rightarrow \langle Q_\gamma, Q_\delta \rangle$  is constrained by the inferential types of the abstracted trees, on the other hand the *ADS-theorem* of  $\Rightarrow \langle Q_\gamma, Q_\delta \rangle$  canonically depends on *ADS-theorems* of the abstracted trees.

The class of modular potential proof trees based on Comp-ADS's of some relevant kind  $K$  has the suitable generality both for the representation of the main properties of a  $\mathbf{LK}_\omega$ -deduction and for the modeling of typed formulas and logical connectives. One of the main challenges this paper tries to meet in the next sections is to select a kind  $K$  of Comp-ADS's including relevant information and, simultaneously, allowing the technical transformations between trees that produce the inferential algebra  $\langle g, \{{}^K\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$ : indeed, the operations of the algebra act essentially on the main Comp-ADS's of the two involved monic trees.

## 4. Inferential Interpretations of typed formulas in the domains of an Inferential Algebra

The aim is to introduce an interpretation  $V$  from the set  $\cup_{\alpha \in \text{types}} \{B_\alpha : B_\alpha \text{ closed formula of type } \alpha\}$  into  $\cup_{\alpha \in \text{types}} {}^K\mathbf{D}_\alpha$  that links each sentence  $B_\alpha$  to a potential proof-tree in  ${}^K\mathbf{D}_\alpha$ . It is desirable that such link is injective, recursive, and such that the potential proof-tree associated to  $B_\alpha$  contains effective information on the structure of  $B_\alpha$ . Moreover, the set of potential proof trees in  $\cup_{\alpha \in \text{types}} {}^K\mathbf{D}_\alpha$  that are associated by  $V$  to the  $\mathbf{LK}_\omega$ -sentences should be closed under  $\Rightarrow, *$ , i.e. it must be an inferential sub-algebra of  $\langle g, \{{}^K\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$ .

**Definition 4.1.** Let  $K$  be a fixed kind of Comp-ADS's, let  $\langle g, \{{}^K\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$  be a fixed inferential algebra (Definition 3.6) and let  $B_\alpha$  be a  $\mathbf{LK}_\omega$ -closed formula; then an *inferential interpretation for  $\mathbf{LK}_\omega$  based on  $K$*  is an injective recursive function:

$$V: \cup_{\alpha \in \text{types}} \{B_\alpha : B_\alpha \text{ closed formula} \} \rightarrow \langle g, \{{}^K\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$$

such that  $V(B_\alpha)$  must be recursively constructable given  $B_\alpha$ , having the following properties:

- i)  $V(B_\alpha) \in {}^K\mathbf{D}_\alpha$ , it is a monic potential proof-tree of  $K$ -inferential type  $\alpha$ , i.e. having a unique main  $K$ -Comp-ADS  $\mathcal{T}$ .
  - ii)  $B_\alpha$  is the canonical component (Def. 3.9.ii) of the theorem  $A_\beta$  of the main Comp-ADS  $\mathcal{T}$  of  $V(B_\alpha)$ .
  - iii) The set  $\cup_{\alpha \in \text{types}} \{V(F_\alpha)\}$  with the operations  $\Rightarrow, *$ , is a *proper sub-algebra* of  $\langle g, \{{}^K\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$ .
- ◇

**Definition 4.2.** An inferential interpretation  $V$  is *functionally sound* if the following properties hold:

- i)  $V(A_{\gamma \rightarrow \delta}) * V(B_\gamma)$  is identical to  $V(A_{\gamma \rightarrow \delta} B_\gamma) \in \{V(F_\delta)\} \subset {}^K\mathbf{D}_\delta$ ;
  - ii)  $V(A_\theta) \Rightarrow V(B_\eta)$  is identical to  $V(g(A_\theta, B_\eta)) \in \{V(F_{\theta \rightarrow \eta})\} \subset {}^K\mathbf{D}_{\theta \rightarrow \eta}$ ,  $g$  canonical bijection of the algebra.
- ◇

**Remark 4.3.** If  $V$  is any inferential interpretation, it can be noted that:

- 1) Each  $V(F_{\gamma \rightarrow \delta})$  can be identified, by application, with a recursive function  $\{V(B_\gamma) : B_\gamma \text{ closed formula}\} \rightarrow \{V(A_\delta) : A_\delta \text{ closed formula}\}$ . The set  $\{V(F_{\gamma \rightarrow \delta}) : F_{\gamma \rightarrow \delta} \text{ closed formula}\}$  is a countable sub-set of the set of the recursive functions  ${}^K\mathbf{D}_\gamma \rightarrow {}^K\mathbf{D}_\delta$ .
- 2) Through  $V$  the sentence  $B_\gamma$  is expressed as an inference having a specific higher order character, that is as an elementary higher order deduction structure. This aspect will be modeled principally by the *main Comp-ADS*  $\mathcal{T}$  of  $V(B_\gamma)$ , the inferential type of which is  $\gamma$ . Moreover, since  $B_\gamma$  explicitly occurs as a leftmost term of the  $\mathcal{T}$ -theorem, the main logical connective of  $B_\gamma$  is the leftmost symbol of the theorem of  $\mathcal{T}$ .
- 3) If  $V$  is functionally sound then it behaves soundly w.r.t. the term application of the language. ◇

**Theorem 4.4 (Main Theorem).** A kind  $K$  of Comp-ADS's is definable such that:

- i) The set of inferential algebras of modular  $\mathbf{LK}_\omega$ -trees  $\langle g, \{{}^K\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$  based on  $K$  is not empty.
- ii) An algebra  $\mathcal{A} \equiv \langle g, \{{}^K\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$  is definable, having a sub-algebra with all the properties required for the image of an inferential interpretation, and such that the set of the functionally sound inferential interpretations  $V$  of the  $\mathbf{LK}_\omega$ -sentences is not empty.

Thus, logical connectives are interpreted in  $\mathcal{A}$  as follows: if  $\star$  is a binary logical connective then  $(\star B_\delta) A_\gamma$  is interpreted by  $(V(\star) * V(B_\delta)) * V(A_\gamma)$ ; if  $\blacktriangledown$  is a monadic logical connective then  $\blacktriangledown B_\delta$  is interpreted by  $V(\blacktriangledown) * V(B_\delta)$ . The interesting fact is that  $V(\star)$  and  $V(\blacktriangledown)$  are potential proof-trees, i.e. that the connective becomes a potential proof. Thus, the intended semantics of connectives in constructive logic becomes here explicit and formal. The proof of Main Theorem will be shown in Section 8, after the definition, in Sections 6 and 7, of the Comp-ADS kind (i.e. the critical chains) and of the inferential algebra that allow the stated results. An example of inferential interpretation is given in Appendix A1.

## 5. Inferential Semantics structures and truth

Inferential algebras and inferential interpretations provide a full semantics for  $\mathbf{LK}_\omega$ , which will be called *inferential semantics*, admitting soundness and completeness theorems. The main notion introduced in this section is that of *inferential structure*, where the truth of an  $\mathbf{LK}_\omega$ -formula is defined. A preliminary notion is that of *inferential frame*, where a semantical identity relation between trees which are interpretations of the form  $V(B_\alpha)$  is established.

**Definition 5.1.** We call *higher order system* any system  $\Delta$  such that: i) the language of  $\Delta$  is an expansion of the  $\mathbf{LK}_\omega$ -language at most through a denumerable set of new primitive non-logical constants for each type  $\gamma$ ; ii) the rule set  $Ru(\Delta)$  of  $\Delta$  is included in that of  $\mathbf{LK}_\omega$ ; iii)  $\Delta$  has a possibly empty set  $Ax(\Delta)$  of proper axioms, and the same logical axioms as  $\mathbf{LK}_\omega$ . ◇

**Definition 5.2.** Let  $\Delta$  be a consistent higher order system. Then, if  $\mathcal{A} \equiv \langle g, \{{}^K\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$  is an inferential algebra and  $G$  an inferential interpretation, an *inferential semantics frame* (briefly *inferential frame*) is a triple  $\mathcal{F} \equiv \langle (\{{}^K\mathbf{D}_\alpha\}, G), \Delta, \cong \rangle$  where  $\cong$  is the *semantical identity relation* established by  $\mathcal{F}$  between those elements of  ${}^K\mathbf{D}_\alpha$  that are images of  $G$ , so that  $G(A_\alpha) \cong G(B_\alpha)$  iff a proof in  $\Delta$  exists of the sentence  $A_\alpha =_\alpha B_\alpha$ .

◇

Therefore, in a broad sense, a frame identifies two interpretation trees if the relevant components of the theorems of their main Comp-ADS's are provably equal in the system of the frame. However, it is meaningless to speak of the truth of a formula in a generic frame  $\mathcal{F}$ , if previously it hasn't been checked that  $\mathcal{F}$  is a *sound denotation* with respect to the logical constants and that sound assignments on variables are possible.

**Definition 5.3.** An inferential frame  $\mathcal{F} \equiv \langle (\{\mathbf{K}\mathbf{D}_\alpha\}, \mathbf{V}), \Delta, \cong \rangle$  is a *sound functional denotation for the logical constants and the equality symbol  $=_\alpha$*  if  $\mathbf{V}$  is functionally sound and the following conditions hold:

- i)  $[\mathbf{V}(\bigwedge_{o \rightarrow (o \rightarrow o)}) * \mathbf{V}(B_o)] * \mathbf{V}(C_o) \cong \mathbf{V}(T)$  iff  $\mathbf{V}(B_o) \cong \mathbf{V}(T)$  and  $\mathbf{V}(C_o) \cong \mathbf{V}(T)$ ;  
 $\mathbf{V}[(\bigwedge_{o \rightarrow (o \rightarrow o)}) * \mathbf{V}(B_o)] * \mathbf{V}(C_o) \cong \mathbf{V}(\perp)$  otherwise.
- ii)  $[\mathbf{V}(\bigvee_{o \rightarrow (o \rightarrow o)}) * \mathbf{V}(B_o)] * \mathbf{V}(C_o) \cong \mathbf{V}(T)$  iff one between  $\mathbf{V}(B_o)$  and  $\mathbf{V}(C_o)$  is  $\cong \mathbf{V}(T)$ ;  
 $[\mathbf{V}(\bigvee_{o \rightarrow (o \rightarrow o)}) * \mathbf{V}(B_o)] * \mathbf{V}(C_o) \cong \mathbf{V}(\perp)$  otherwise.
- iii)  $\mathbf{V}(\neg_{o \rightarrow o}) * \mathbf{V}(A_o) \cong \mathbf{V}(T)$  iff  $\mathbf{V}(A_o) \cong \mathbf{V}(\perp)$ .
- iv)  $[\mathbf{V}(\bigcup_{o \rightarrow (o \rightarrow o)}) * \mathbf{V}(B_o)] * \mathbf{V}(C_o) \cong \mathbf{V}(T)$  iff either  $\mathbf{V}(B_o) \cong \mathbf{V}(\perp)$  or  $\mathbf{V}(C_o) \cong \mathbf{V}(T)$ ;  
 $[\mathbf{V}(\bigcup_{o \rightarrow (o \rightarrow o)}) * \mathbf{V}(B_o)] * \mathbf{V}(C_o) \cong \mathbf{V}(\perp)$  otherwise.
- v)  $\mathbf{V}(\forall_{(\alpha \rightarrow o) \rightarrow o}) * \mathbf{V}(\lambda x_\alpha B_o) \cong \mathbf{V}(T)$ , iff  $\mathbf{V}(\lambda x_\alpha B_o) * \mathbf{V}(h_\alpha) \cong \mathbf{V}(T)$  for each closed formula  $h_\alpha$  of the  $\Delta$ -language;  
 $\mathbf{V}(\forall_{(\alpha \rightarrow o) \rightarrow o}) * \mathbf{V}(\lambda x_\alpha B_o) \cong \mathbf{V}(\perp)$  otherwise.
- vi)  $\mathbf{V}(\exists_{(\alpha \rightarrow o) \rightarrow o}) * \mathbf{V}(\lambda x_\alpha B_o) \cong \mathbf{V}(T)$ , iff  $\mathbf{V}(\lambda x_\alpha B_o) * \mathbf{V}(h_\alpha) \cong \mathbf{V}(T)$  for at least a closed formula  $h_\alpha$  of the  $\Delta$ -language;  
 $\mathbf{V}(\exists_{(\alpha \rightarrow o) \rightarrow o}) * \mathbf{V}(\lambda x_\alpha B_o) \cong \mathbf{V}(\perp)$  otherwise.
- vii)  $[\mathbf{V}(\equiv_{\alpha \rightarrow (\alpha \rightarrow o)}) * \mathbf{V}(A_\alpha)] * \mathbf{V}(B_\alpha) \cong \mathbf{V}(T)$  iff  $\mathbf{V}(A_\alpha) \cong \mathbf{V}(B_\alpha)$ .

◇

The soundness of the logical constants  $\perp, T$ , holds in each frame  $\langle (\{\mathbf{K}\mathbf{D}_\alpha\}, \mathbf{V}), \Delta, \cong \rangle$ , since  $\mathbf{V}(\perp) \cong \mathbf{V}(T)$  is never possible due to the consistency of  $\Delta$ .

**Definition 5.4.** Given an inferential frame  $\mathcal{F} \equiv \langle (\{\mathbf{K}\mathbf{D}_\alpha\}, \mathbf{V}), \Delta, \cong \rangle$ , an *assignment on the free variables associated to  $\mathcal{F}$*  is a function  $\varphi : \{\text{free variables of } \mathbf{LK}_\omega\text{-language}\} \rightarrow \cup_\alpha \mathbf{K}\mathbf{D}_\alpha$  such that  $\varphi(a_\gamma) \in \mathbf{K}\mathbf{D}_\gamma \cap \text{Im}\mathbf{V}$ <sup>6</sup>. A  $\varphi$ -variant is an assignment  $\psi$  coinciding with  $\varphi$ , with the exclusion of a fixed  $b_\alpha$  such that  $\varphi(b_\alpha)$  is different from  $\psi(b_\alpha)$ . ◇

$\varphi(a_\gamma)$  is therefore a potential proof-tree  $\mathbf{V}(C_\gamma)$  for some closed formula  $C_\gamma$ .

**Definition 5.5.** Let  $\mathcal{F} \equiv \langle (\{\mathbf{K}\mathbf{D}_\alpha\}, \mathbf{V}), \Delta, \cong \rangle$  be an inferential frame, and  $\varphi$  an assignment on the free variables associated to it. Then, an *interpretation function of the typed formulas based on the pair  $(\mathcal{F}, \varphi)$* , is a function  $\mathbf{V}_\varphi : \{F_\alpha : \alpha \text{ a type}\} \rightarrow \cup \mathbf{K}\mathbf{D}_\gamma$ , so that  $\mathbf{V}_\varphi(H_\gamma) \in \mathbf{K}\mathbf{D}_\gamma$ , which is defined as follows:  $\mathbf{V}_\varphi(F_\beta)$  is  $\mathbf{V}(F_\beta)$  iff  $F_\beta$  is closed,  $\mathbf{V}_\varphi(A_\beta)$  is  $\mathbf{V}(A_\beta^\varphi)$  iff  $A_\beta$  is open,  $A_\beta^\varphi$  being the closed formula obtained from  $A_\beta$  by replacing each free variable  $a_\gamma$  with the unique closed formula  $C_\gamma$  such that  $\varphi(a_\gamma) = \mathbf{V}(C_\gamma)$ . ◇

Note that, by definition, for each free variable  $a_\beta$ ,  $\mathbf{V}_\varphi(a_\beta) = \varphi(a_\beta)$ . The soundness of the denotation w.r.t. the  $\lambda$ -abstraction of variables must also be imposed, getting the notion of inferential structure:

**Definition 5.6.** Let  $\mathcal{M} \equiv \langle (\{\mathbf{K}\mathbf{D}_\alpha\}, \mathbf{V}), \Delta, \cong \rangle$  be an inferential frame which is a sound functional denotation for the logical constants and the equality symbol  $=_\alpha$ . For each assignment  $\varphi$  on the free variables associated to  $\mathcal{M}$ , let  $\mathbf{V}_\varphi$  be the interpretation based on the pair  $(\mathcal{M}, \varphi)$ ; then  $\mathcal{M}$  is an *inferential structure based on the algebra  $\mathcal{A} \equiv \langle g, \{\mathbf{K}\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$* , if the following condition holds:  $\mathbf{V}_\varphi(\lambda x_\alpha A_\beta)$  is the potential proof-tree  $P \in \mathbf{K}\mathbf{D}_{\alpha \rightarrow \beta}$  such that, for each closed  $\alpha$ -typed formula  $F_\alpha$ ,  $P * \mathbf{V}(F_\alpha) \cong \mathbf{V}_\psi(A_\beta) \in \mathbf{K}\mathbf{D}_\alpha$ , where  $\psi$  is the  $\varphi$ -variant of  $\varphi$  which maps the free variable  $b_\alpha$  corresponding to  $x_\alpha$  and not occurring in  $\lambda x_\alpha A_\beta$ , into  $\psi(b_\alpha) \equiv \mathbf{V}(F_\alpha)$ . ◇

Inside an inferential structure the notion of truth can be defined:

**Definition 5.7.** Let  $\mathcal{M} \equiv \langle (\{\mathbf{K}\mathbf{D}_\alpha\}, \mathbf{V}), \Delta, \cong \rangle$  be an inferential structure. Then, a closed  $o$ -typed formula  $A$  is *true* in  $\mathcal{M}$  iff  $\mathbf{V}(A) \cong \mathbf{V}(T)$ , it is *false* iff  $\mathbf{V}(A) \cong \mathbf{V}(\perp)$ . An open  $o$ -typed formula  $B$  is *true* with respect to an interpretation  $\mathbf{V}_\varphi$  based on the pair  $(\mathcal{M}, \varphi)$  iff  $\mathbf{V}_\varphi(B) \equiv \mathbf{V}(T)$ , it is false iff  $\mathbf{V}_\varphi(B) \equiv \mathbf{V}(\perp)$ .

A closed  $o$ -typed formula  $A$  is *possibly true* in  $\mathcal{M}$  if neither  $\mathbf{V}(A) \cong \mathbf{V}(T)$  nor  $\mathbf{V}(A) \cong \mathbf{V}(\perp)$  holds.

◇

<sup>6</sup>As usual  $\text{Im}\mathbf{V}$  stands for the image of the function  $\mathbf{V}$ .

Note that sentences  $A$  possibly true in  $\mathcal{M}$  are such that both  $A$  and  $\neg A$  are  $\Delta$ -consistent. We will not develop a multivalued semantics inside inferential structures in this paper, and as to the completeness result (Section 9) we will consider structures  $\langle \{\mathbf{K}\mathbf{D}_\alpha\}, \mathbf{V}, \Delta, \cong \rangle$  where  $\Delta$  is a syntactically complete system. However, such three-valued denotation could have a role in the inferential semantics for constructive logics, which is a work in progress (Section 11).

**Definition 5.8.** Given a consistent system  $\Lambda$  extending  $\mathbf{LK}_\omega$  and an inferential structure  $\mathcal{M} \equiv \langle \{\mathbf{K}\mathbf{D}_\alpha\}, \mathbf{V}, \Delta, \cong \rangle$ , such that  $\Delta$  and  $\Lambda$  have the same language,  $\mathcal{M}$  is an *inferential model* of  $\Lambda$  iff the  $\Lambda$ -theorems are true in  $\mathcal{M}$ .

◇

It must be remarked that  $\Delta \vdash B =_o \top$  is a necessary but not a sufficient condition for the truth of  $B$  in the inferential structure  $\mathcal{M} \equiv \langle \{\mathbf{K}\mathbf{D}_\alpha\}, \mathbf{V}, \Delta, \cong \rangle$ . The central requirements are the properties of the inferential algebra  $\mathcal{A} \equiv \langle g, \{\mathbf{K}\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$  that allow  $\mathcal{M}$  to be a sound denotation. For example, a frame of the form  $\langle \{\mathbf{K}\mathbf{D}_\alpha\}, \mathbf{V}, \mathbf{LK}_\omega, \cong \rangle$  is not, in general, an inferential model of  $\mathbf{LK}_\omega$ .

**Theorem 5.9 (Inferential Semantics Theorem).** A kind  $K$  of Comp-ADS's is definable such that:

- i) The class of inferential structures based on the algebra  $\mathcal{A} \equiv \langle g, \{\mathbf{K}\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$  is not empty.
- ii)  $\mathbf{LK}_\omega$  is sound and complete with respect to the class of inferential structures based on  $\mathcal{A} \equiv \langle g, \{\mathbf{K}\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$ .

The proof of Theorem 5.9 above is presented in Section 9.

Finally, a sound and complete inferential semantics for functional type theory, shows that each functional typed formula is itself a kind of potential higher order inference, which includes Comp-rules; simultaneously, each formula can be seen as a recursive function from proofs to proofs.

## 6. The critical chain abstract deduction structure

The theses of Theorems 4.4 and 5.9 establish the main goals of the paper, and the sections from 6 to 9 will provide the proofs of such theorems. To this end, in this Section, a particular kind of Comp-ADS in a  $\mathbf{LK}_\omega$ -tree will be defined: the *critical chain* ADS.<sup>7</sup> It is useful to choose as elements of a Comp-ADS terms which are connected by a logical link that takes into account the ancestor-descendant relation in a proof. It can be observed that the chains of formulas connected by a suitable ancestor-descendant relation on a branch of a tree, can sometimes better express the inference content of a proof; this method has played a central role in the proof-theoretic demonstration of the arithmetical completeness of modal logic by Gentilini [11, 12, 13]. Nevertheless, in the  $\mathbf{LK}_\omega$  setting, there are different possible ancestor-descendant relations among formulas in a proof  $P$ . That is, the definition of the relation used for the first order logic trees (see, e.g., [27] p. 78) is not sufficient for  $\mathbf{LK}_\omega$ -trees. In  $\mathbf{LK}_\omega$  there are the following new situations:

a) The distinction between terms and formulas, that characterizes the standard first order setting, disappears; moreover, no constraints are given for the position of an  $o$ -typed formula in a context in a tree; then, there is the further notion of *isolated formula* in an  $\mathbf{LK}_\omega$ -sequent (Section 2.3) that allows an ancestor descendant relation which links formulas of type  $o$  that also are isolated formulas (Definition 2.8). This is the *isolated ancestor-descendant relation*, (*isolated a.d. relation* Definition 6.5), which expresses the action of propositional logical rules, and that mutually connects the auxiliary and principal propositions (Definition 2.9) of a rule occurrence.

b) However, also an ancestor descendant relation *between arbitrary terms* must be specified, and, in the most general case, it will link terms having different types. In this relation the type  $o$  will not have any particular status. This is the *term ancestor-descendant relation* (*term a.d. relation* Definition 6.8), which expresses the action of the Comp-rules and of the  $\lambda$ -rule.

Both relations are necessary in order to select Comp-Abstract Deduction Structures that express the relevant inference performed in a proof. In this section these two different relations will be defined.<sup>8</sup>

<sup>7</sup>In Section 6 all the presented notions are new. In order to help the reader to preserve an intuitive control of the main line of the paper, we suggest the following approach: go directly to Def. 6.14 of *critical chain*, which is the key definition of the section. Some details will not be clear; however, a fundamental point can be noticed, i.e. that in the proof-theoretic examination of higher order sequent trees, two different *ancestor-descendant relations* are necessary: the first among *isolated formulas*, the second among *terms*. Then, go directly to Examples 6.17 and 6.18, where some critical chains occurring in higher order trees are effectively presented, and intuitively focus what are the peculiar properties of such ADS kind. At this point, the first part of the section, where a fine analysis of higher order proofs is developed to get the definitions of the two different ancestor-descendant relations mentioned above, should acquire a stronger motivation.

<sup>8</sup>An heuristic example is the following:

$$\frac{X \vdash Y, A_{o \rightarrow o}(B_{\alpha \rightarrow o} C_\alpha)}{X \vdash Y, \exists x_o A_{o \rightarrow o}(x_o)} \exists\text{-R}$$

**Definition 6.1.** [Auxiliary and principal terms in a rule occurrence] Let  $\mathcal{R}$  be a logical rule occurrence (Section 2.4), in an  $\text{LK}_o$ -proof P. Then:

i) If  $\mathcal{R}$  belongs to the propositional logical rules, an *auxiliary term* of the rule is any formula-occurrence which is a sub-term of an auxiliary proposition (Definition 2.9) of the rule, and a *principal term* of the rule is any formula-occurrence which is a sub-term of the principal proposition (Definition 2.9) of the rule. In  $\vee$ -R and  $\wedge$ -L rules, the non isolated  $o$ -typed formula introduced as maximal disjunct (conjunct) in the principal proposition is called the *maximal non isolated introduced term* of the rule and belongs to the set of the principal terms of the rule.

ii) If  $\mathcal{R}$  is a  $\lambda$ -rule, an *auxiliary term* of  $\mathcal{R}$  is any term occurrence in any  $\mathcal{R}$ -auxiliary proposition in the premise, that is included in a  $\beta$ -redex or  $\beta$ -contractum which is reduced or replaced by  $\mathcal{R}$ ; a *principal term of  $\mathcal{R}$*  is any term occurrence in any  $\mathcal{R}$ -principal proposition in the conclusion, that is included in a  $\beta$ -redex or  $\beta$ -contractum which is produced by  $\mathcal{R}$ ; a *maximal auxiliary term* is any auxiliary term not included in any different auxiliary term, and a *maximal principal term* is any principal term not included in any different principal term.

iii) If  $\mathcal{R}$  is a logical rule for quantifiers  $\forall$ -R,  $\exists$ -L, an *auxiliary term of  $\mathcal{R}$*  is any occurrence of the eigenvariable  $b_\alpha$  in the premise, and a *principal term of  $\mathcal{R}$*  is any corresponding occurrence of the bound variable  $x_\alpha$  in the principal proposition of  $\mathcal{R}$  in the conclusion.

iv) If  $\mathcal{R}$  is a Comp-rule, a *maximal auxiliary term* of  $\mathcal{R}$  is any occurrence of the formula  $t_\alpha$  on which the quantification acts in the premise, and a *principal term of  $\mathcal{R}$*  each corresponding occurrence of the bound variable  $x_\alpha$  in the principal proposition of  $\mathcal{R}$  in the conclusion. An *auxiliary term of  $\mathcal{R}$*  is any occurrence of a formula as a sub-term in a maximal auxiliary term in the premise.  $\diamond$

**Remark 6.2.** i) Auxiliary and principal terms of each rule may range among formulas of arbitrary type. Notice that the expressions “auxiliary term” and “principal term” always indicate a term occurrence in a given context, and not simply a term.

ii) Each auxiliary term occurs in an auxiliary proposition and each principal term occurs in a principal proposition; however, if the rule is not a propositional logical rule, the set of auxiliary or principal terms is in general a proper subset of the sub-terms of the respective auxiliary or principal propositions (e.g., this is the case of quantifier rules).

iii) If  $t_\alpha$  is a maximal auxiliary term in a Comp-rule premise  $S$ , it may however happen that in  $S$  other occurrences of  $t_\alpha$  exist that are not auxiliary terms of the rule. Moreover, the set of the different terms and types which occur as auxiliary terms in the maximal auxiliary  $t_\alpha$ , may be arbitrarily large. Conversely, each principal term in a Comp-rule is an atom.

$\diamond$

**Example 6.3.** Consider the following proof P:

$$\frac{\frac{\frac{z_o \vdash z_o}{z_o \vdash \exists x_o(x_o)} \exists\text{-R} \quad A_{o \rightarrow o} y_o \vdash A_{o \rightarrow o} y_o}{A_{o \rightarrow o} y_o, z_o \vdash \exists x_o(x_o) \wedge A_{o \rightarrow o} y_o} \wedge\text{-R}}{A_{o \rightarrow o} y_o, z_o \vdash \exists u_o(\exists x_o(x_o) \wedge u_o)} \exists\text{-R}$$

where we recall that the writing  $\exists x_o(x_o)$  is an abbreviation for  $\exists_{(o \rightarrow o) \rightarrow o} \lambda x_o x_o$ . The maximal auxiliary term of the uppermost  $\exists$ -R is  $z_o$ , which is the only auxiliary term, and the principal term is  $x_o$ ; note that the auxiliary term is in this case the auxiliary proposition too, while the principal term is not a proposition of the rule occurrence; the set of the auxiliary terms of the lowermost  $\exists$ -R is:  $\{A_{o \rightarrow o} y_o(\text{maximal}), A_{o \rightarrow o}, y_o\}$ ; the principal term is  $u_o$ ; the auxiliary and principal propositions of the rule occurrence are respectively  $\exists x_o(x_o) \wedge A_{o \rightarrow o} y_o$  and  $\exists u_o(\exists x_o(x_o) \wedge u_o)$ ;

the set of the auxiliary terms of the  $\wedge$ -R occurrence is:  $\{A_{o \rightarrow o} y_o, A_{o \rightarrow o}, y_o, \exists x_o(x_o), x_o, \exists_{(o \rightarrow o) \rightarrow o} \lambda x_o x_o\}$ ; the set of the principal terms is:  $\{A_{o \rightarrow o} y_o, A_{o \rightarrow o}, y_o, \exists x_o(x_o), x_o, \exists_{(o \rightarrow o) \rightarrow o} \lambda x_o x_o, \exists x_o(x_o) \wedge A_{o \rightarrow o} y_o, \bigwedge_{o \rightarrow (o \rightarrow o)}\}$ .  $\diamond$

First, an ancestor-descendant relation among isolated formulas will be defined, that connects the propositions of the rules, and that is closer to the expression of the propositional logical inference of the proof P; then, it will be defined an ancestor-descendant relation among arbitrary formulas, involving auxiliary and principal terms of the rules, which is closer to the expression of the Comp-rule inference of the proof P. The concept of *critical chain* will mix the two relations in order to have a selected complete representation of the inferential content of P.

---

The isolated formula  $A_{o \rightarrow o}(B_{\alpha \rightarrow o} C_\alpha)$  in the premise has the isolated formula  $\exists x_o A_{o \rightarrow o}(x_o)$  in the conclusion as descendant in the isolated a.d. relation; but its descendant in the term a.d. relation is  $A_{o \rightarrow o}(x_o)$ . The formula occurrences  $B_{\alpha \rightarrow o} C_\alpha, B_{\alpha \rightarrow o}, C_\alpha$  in the premise, all have the same  $x_o$  occurrence in the argument of  $A_{o \rightarrow o}$  in the conclusion as term descendant; they have not an isolated a.d. descendant, since they cannot be the elements of an isolated a.d. relation. Conversely,  $\exists x_o A_{o \rightarrow o}(x_o)$  in the conclusion, has no term ancestors in the premise, since the auxiliary and the principal proposition of this rule cannot be connected by a term a.d. relation. In this section we will also consider the terms  $B_{\alpha \rightarrow o}$  and  $x_o$ , despite having different types, to be connected by a kind of inference relation

**Definition 6.4.** [isolated predecessor, isolated successor ]

i) Let  $\mathcal{R}$  be a rule occurrence in a  $\mathbf{LK}_\omega$ -proof P; then :

i1) Any auxiliary proposition of  $\mathcal{R}$  in a premise is an *isolated predecessor* of the corresponding principal proposition of  $\mathcal{R}$ , which is its *isolated successor*;

i2) Any isolated formula occurrence in a premise, that is not an  $\mathcal{R}$ -auxiliary proposition, is an *isolated predecessor* of the isolated formula occurrence in the conclusion corresponding to it, which is its *isolated successor*; in this case predecessor and successor are occurrences of the same isolated formula.

ii) If  $\mathcal{R}$  is a Cut-rule or a weakening rule the notion of *isolated predecessor* and of *isolated successor* are defined as in the point i2) for all the isolated formulas occurrences which are not a cut-formula or the weakening-formula.

◇

**Definition 6.5.** [ancestor-descendant relation among isolated formula occurrences (Section 2.8)]

i) Two isolated formula occurrences  $A$  and  $B$  in P are *linked by an ancestor-descendant relation between isolated formulas* (briefly, *isolated a.d. relation*) if they are respectively the top and the end of a sequence of isolated formula occurrences in P, mutually connected by an isolated predecessor-isolated successor relation.  $A$  is an *isolated ancestor* of  $B$  in P, and  $B$  is a *isolated descendant* of  $A$  in P.

ii) The *isolated a.d. chain of extremes*  $A, B$  in P is the mentioned sequence of isolated formula occurrences between  $A$  and  $B$ .  $B$  is an *integral isolated descendant* of  $A$  (and  $A$  is an *integral isolated ancestor* of  $B$ ) if no element in the chain is the proposition of a rule; in this case  $A$  and  $B$  are occurrences of the same  $o$ -typed formula.

◇

**Example 6.6.** In the proof P of example 6.3 the following sequence of  $o$ -typed occurrences:

$$\langle z_o, \exists x_o(x_o), \exists x_o(x_o) \wedge A_{o \rightarrow o} y_o, \exists u_o(\exists x_o(x_o) \wedge u_o) \rangle$$

is an isolated a. d. chain of extremes  $z_o$  and  $\exists u_o(\exists x_o(x_o) \wedge u_o)$ . ◇

**Definition 6.7.** [term predecessor, term successor]

i) Let  $\mathcal{R}$  be a rule occurrence in a  $\mathbf{LK}_\omega$ -proof P; then:

i.1) Any auxiliary term (Def. 6.1) of  $\mathcal{R}$  in a premise is a *term predecessor* of a corresponding principal term of  $\mathcal{R}$ , which is its *term successor*, with the following specifications:

- if  $\mathcal{R}$  is a Comp-rule, any maximal (Definition 6.1) auxiliary term  $t_\alpha$  of  $\mathcal{R}$  is a *maximal term predecessor* of the principal term  $x_\alpha$  of  $\mathcal{R}$ , which is its *term successor*; *term predecessors* of  $x_\alpha$  are all the proper sub-term occurrences in its maximal term predecessor  $t_\alpha$ ;
- if  $\mathcal{R}$  is a propositional logical rule, any pair of formula occurrences connected by a term predecessor-term successor relation, must be a pair of occurrences of the same formula, and each formula in the premise has exactly one term successor in the conclusion. Conversely, any principal term in the conclusion may have no term-predecessor in the premise; in particular, in each logical rule, the introduced logical constant occurring in the principal proposition has no term predecessors, and the principal proposition of the rule has no term predecessors. In the rules  $\wedge$ -L and  $\vee$ -R, the maximal non isolated introduced term (Def. 6.1.i) has no term predecessors.
- if  $\mathcal{R}$  is a  $\lambda$ -rule, the *successor* of each maximal auxiliary term is the corresponding maximal principal term ;  
the *successor* of each non maximal auxiliary term  $t$ , such that no term occurrences of the form  $t$  appear in the conclusion of the rule, is the leftmost term of the corresponding maximal principal term;  
the *successors* of each auxiliary term  $t$  such that term occurrences of the form  $t$  appear in the conclusion of the rule, are all the occurrences of the form  $t$  in the corresponding maximal principal term;  
if to any non maximal principal term  $r$ , no occurrences of the form  $r$  correspond in the maximal auxiliary term linked to it by the rule, then  $r$  has *no term predecessors*.

i.2) Any formula occurrence in a premise, that is not a  $\mathcal{R}$ -auxiliary term, is a *term predecessor* of the corresponding formula occurrence in the conclusion, which is its *term successor*; in this case predecessors and successor are occurrences of the same formula.

ii) If  $\mathcal{R}$  is a Cut-rule or a weakening rule in P, the notion of *term predecessor* and of *term successor* are defined as in the point i.2) for all the formulas which do not occur in a cut-formula or in a weakening-formula.

◇

**Definition 6.8.** [ancestor-descendant relation among term occurrences, i.e. arbitrary formula occurrences]

i) Two term occurrences  $C_\beta$  and  $F_\gamma$  in  $P$  are linked by an ancestor-descendant relation between terms (briefly, term a.d. relation) if they are respectively the top and the end of a sequence of formula occurrences in  $P$ , mutually connected by a term predecessor-term successor relation.  $C_\beta$  is a term ancestor of  $F_\gamma$  in  $P$ , and  $F_\gamma$  is a term descendant of  $C_\beta$  in  $P$ .

ii) The term a.d. chain of extremes  $C_\beta, F_\gamma$  in  $P$  is the mentioned sequence of formula occurrences between  $C_\beta$  and  $F_\gamma$ .  $F_\gamma$  is an integral term descendant of  $C_\beta$  (and  $C_\beta$  is an integral term ancestor of  $F_\gamma$ ) if no element in the chain (possibly except  $F_\gamma$ ) is an auxiliary term of a rule occurrence  $\mathcal{R}$  with  $\mathcal{R}$  a quantifier logical rule, or a  $\lambda$ -rule deleting the  $C_\beta$ -term descendants having the same form of  $C_\beta$ ; in this case  $C_\beta$  and  $F_\gamma$  are occurrences of the same formula. A term a.d. chain of extremes  $C_\beta, F_\gamma$  is maximal in  $P$  if  $C_\beta$  has no term predecessors and  $F_\gamma$  has no term successors.

◇

**Example 6.9.** Consider the following  $\vee$ -R rule occurrence:

$$\frac{Y \vdash \Delta, F}{Y \vdash \Delta, F \vee B} \vee\text{-R}$$

the maximal non isolated introduced term of the rule is  $B$ , occurring in the principal proposition  $F \vee B$ , and it has no term predecessors. ◇

**Example 6.10.** Consider the following proof-segment  $\mathbf{H}$ :

$$\frac{\frac{X \vdash \lambda x_\alpha (A_{\alpha \rightarrow o} x_\alpha \vee B_{\alpha \rightarrow o} x_\alpha) M_\alpha}{X \vdash A_{\alpha \rightarrow o} M_\alpha \vee B_{\alpha \rightarrow o} M_\alpha} \lambda_1}{X \vdash A_{\alpha \rightarrow o} M_\alpha \vee \lambda y_\alpha (B_{\alpha \rightarrow o} y_\alpha) M_\alpha} \lambda_2$$

In  $\lambda_1$  the  $\beta$ -reduction operates downwards, in  $\lambda_2$  upwards. Furthermore: the auxiliary proposition of  $\lambda_1$  is  $\lambda x_\alpha (A_{\alpha \rightarrow o} x_\alpha \vee B_{\alpha \rightarrow o} x_\alpha) M_\alpha$  which is the maximal auxiliary term too; the principal proposition of  $\lambda_1$  is  $A_{\alpha \rightarrow o} M_\alpha \vee B_{\alpha \rightarrow o} M_\alpha$  which is the maximal principal term too. The term successors of the  $\lambda_1$ -auxiliary term  $M_\alpha$  in the  $\lambda_1$ -premise are both the occurrences of  $M_\alpha$  in the conclusion; the  $\lambda_1$ -auxiliary terms  $\lambda x_\alpha (A_{\alpha \rightarrow o} x_\alpha \vee B_{\alpha \rightarrow o} x_\alpha)$ ,  $A_{\alpha \rightarrow o} x_\alpha$ ,  $B_{\alpha \rightarrow o} x_\alpha$ ,  $x_\alpha$ , are deleted by the rule and their term successor is the outermost term  $\bigvee_{o \rightarrow (o \rightarrow o)}$  of the corresponding  $\lambda_1$ -maximal principal term. The maximal auxiliary term of  $\lambda_2$  is  $B_{\alpha \rightarrow o} M_\alpha$  and its term successor is the  $\lambda_2$ -maximal principal term  $\lambda y_\alpha (B_{\alpha \rightarrow o} y_\alpha) M_\alpha$ ; the non maximal  $\lambda_2$ -principal terms  $\lambda y_\alpha (B_{\alpha \rightarrow o} y_\alpha)$ ,  $B_{\alpha \rightarrow o} y_\alpha$  and  $y_\alpha$ , introduced by the rule, have no term predecessors; the term predecessors of the non maximal  $\lambda_2$ -principal terms  $B_{\alpha \rightarrow o}, M_\alpha$  in the conclusion are, respectively, the corresponding occurrences of  $B_{\alpha \rightarrow o}, M_\alpha$  in the premise. Both occurrences of  $M_\alpha$  in the root are integral descendants of the  $M_\alpha$ -occurrence in the  $\lambda_1$ -premise. Moreover:  $\langle M_\alpha, M_\alpha, M_\alpha \rangle$ , where all the considered  $M_\alpha$ -occurrences are the leftmost in the respective sequents, and  $\langle \lambda x_\alpha (A_{\alpha \rightarrow o} x_\alpha \vee B_{\alpha \rightarrow o} x_\alpha), \bigvee_{o \rightarrow (o \rightarrow o)}, \bigvee_{o \rightarrow (o \rightarrow o)} \rangle$ , are term a.d. chains in  $\mathbf{H}$ . ◇

**Example 6.11.** Consider the following proof  $P$ :

$$\frac{\frac{\frac{z_o \vdash z_o}{z_o \vdash \exists x_o (x_o)} \exists\text{-R} \quad A_{o \rightarrow o} y_o \vdash A_{o \rightarrow o} y_o}{A_{o \rightarrow o} y_o, z_o \vdash \exists x_o (x_o) \wedge A_{o \rightarrow o} y_o} \wedge\text{-R}}{A_{o \rightarrow o} y_o, z_o \vdash \exists u_o (\exists x_o (x_o) \wedge u_o)} \exists\text{-R}}{\frac{A_{o \rightarrow o} y_o, z_o \vdash \exists w_{(o \rightarrow o) \rightarrow o} (w_{(o \rightarrow o) \rightarrow o} \lambda u_o (\exists x_o (x_o) \wedge u_o))}{\neg \exists w_{(o \rightarrow o) \rightarrow o} (w_{(o \rightarrow o) \rightarrow o} \lambda u_o (\exists x_o (x_o) \wedge u_o)), A_{o \rightarrow o} y_o, z_o \vdash} \neg\text{-L}}{\forall h_o (h_o), A_{o \rightarrow o} y_o, z_o \vdash} \forall\text{-R}}$$

j) the following sequences of formula occurrences are term a.d. chains in  $P$ :

$\langle z_o, x_o, x_o, x_o, x_o, x_o, h_o \rangle$  starting from the succedent of the leftmost axiom; it is a maximal chain;

$\langle A_{o \rightarrow o}, A_{o \rightarrow o}, u_o, u_o, u_o, h_o \rangle$  starting from the outermost term of the succedent of the rightmost axiom; it is a maximal chain;

$\langle \exists_{(o \rightarrow o) \rightarrow o}, w_{(o \rightarrow o) \rightarrow o}, w_{(o \rightarrow o) \rightarrow o}, h_o \rangle$  starting from the outermost term of the principal proposition of the second  $\exists$ -R downwards; it is a maximal chain;

$\langle \neg \exists w_{(o \rightarrow o) \rightarrow o} (w_{(o \rightarrow o) \rightarrow o} \lambda u_o (\exists x_o (x_o) \wedge u_o)), h_o \rangle$  starting from the principal proposition (which is obviously the maximal principal term too) of the  $\neg$ -L rule; it is a maximal chain since the principal proposition of a logical rule does not have term predecessors.

Note that: the occurrence of  $A_{o \rightarrow o}$  in the root is an integral term-descendant of the occurrence of  $A_{o \rightarrow o}$  in the antecedent of the rightmost axiom; in a term a.d. chain the type is not preserved, and this is essentially due to Comp-rules action.

jj) the following are isolated a.d. chains in P:

$\langle z_o, \exists x_o(x_o), \exists x_o(x_o) \wedge A_{o \rightarrow o}y_o, \exists u_o(\exists x_o(x_o) \wedge u_o), \exists w_{(o \rightarrow o) \rightarrow o}(w_{(o \rightarrow o) \rightarrow o} \lambda u_o(\exists x_o(x_o) \wedge u_o)), \neg \exists w_{(o \rightarrow o) \rightarrow o}(w_{(o \rightarrow o) \rightarrow o} \lambda u_o(\exists x_o(x_o) \wedge u_o)), \forall h_o(h_o) \rangle$  starting from the succedent of the leftmost axiom and ending at the leftmost isolated formula in the root; this cannot also be a term a.d. chain.

$\langle A_{o \rightarrow o}y_o, A_{o \rightarrow o}y_o, A_{o \rightarrow o}y_o, A_{o \rightarrow o}y_o, A_{o \rightarrow o}y_o \rangle$  starting from the occurrence of  $A_{o \rightarrow o}y_o$  in the antecedent of the rightmost axiom; this is also a term a.d. chain in P.  $\diamond$

In order to define the notion of critical chain, a distinction between propositional inferences has to be done: to include in a critical chain only the logical propositional inference which is the farthest from weakening rules, i.e. having the uppermost ancestors of its auxiliary propositions introduced by axioms and not by weakenings.

**Proposition 6.12.** *Let  $\mathcal{R}$  be an occurrence of a rule with  $\mathcal{R} \in \{\vee\text{-L}, \wedge\text{-R}, \supset\text{-L}, \neg\text{-R}, \neg\text{-L}\}$  in a cut-free  $\mathbf{LK}_\omega$ -proof  $P$  of a sequent  $S$ , and let  $F$  be the principal proposition of  $\mathcal{R}$ . Let the sequent  $S'$ , obtained from  $S$  by deleting  $F$ , be not  $\mathbf{LK}_\omega$ -provable. Then  $F$  has in  $P$  at least two uppermost isolated ancestors occurring in two axioms, if  $\mathcal{R}$  introduces a binary connective, or one uppermost isolated ancestor occurring in one axiom, if  $\mathcal{R}$  introduces a monadic connective.*

*Proof:* For example let  $\mathcal{R}$  have this form:

$$\frac{A, \Gamma \vdash \Delta \quad B, \Theta \vdash \Omega}{A \vee B, \Theta, \Gamma \vdash \Delta, \Omega} \vee\text{-L}$$

It is straightforward to show that if all the uppermost isolated ancestors of one of the auxiliary propositions  $A, B$  are introduced by weakenings, then a sub-sequent of  $\Theta, \Gamma \vdash \Delta, \Omega$  would be provable, which is a contradiction.  $\diamond$

Conversely, it is possible to produce several examples which show that the thesis of Proposition 6.12 cannot hold for the logical propositional rules  $\vee\text{-R}, \wedge\text{-L}, \supset\text{-R}$ . For  $\vee\text{-R}, \wedge\text{-L}$ , the simplest example is the one where the premise is a logical axiom; for  $\supset\text{-R}$ , an example is the one where a logical axiom is the premise of a weakening introducing a left auxiliary proposition of a  $\supset\text{-R}$  applied to its succedent. Note that the essence of the distinction is the affinity with the weakening rule.

**Definition 6.13.** Let  $\vee\text{-L}, \wedge\text{-R}, \supset\text{-L}, \neg\text{-R}, \neg\text{-L}$  be called the *strong propositional rules* and  $\vee\text{-R}, \wedge\text{-L}, \supset\text{-R}$  the *weak ones*. In an isolated a.d. chain in a proof P,  $B$  is called a *strong isolated descendant of A* if no element between  $A$  and  $B$  in the chain (possibly except  $A$ ) is the principal proposition of a weak propositional rule.  $B$  is a *weak isolated descendant of A* otherwise.  $\diamond$

At this point all the ingredients are ready for the definition of the following abstract deduction structure in an  $\mathbf{LK}_\omega$ -proof:

**Definition 6.14.** [*critical chain*] Let P be a fixed  $\mathbf{LK}_\omega$ -proof tree. A *critical chain* in P is a Comp-ADS (Definition 3.1)  $\mathcal{T}$  with the following properties:

i) Consider  $\mathcal{T}$  as the sequence of ordered pairs  $\langle (A^1, B^1), \dots, (A^m, B^m) \rangle$  (Def. 3.1.i.2), with  $(A^j, B^j)$  occurring in the element  $\mathcal{R}_j$  of the set  $\mathbf{R}$  of Comp-rule occurrences  $\{\mathcal{R}_1, \dots, \mathcal{R}_m\}$ ,  $m \geq 1$  (Def. 3.1.i.1);

ii) For each  $(A^j, B^j)$ ,  $A^j$  is a maximal auxiliary term of a Comp-rule occurrence in P, and  $B^j$  is the principal term (Def. 6.1) which is the successor (Def. 6.7) of  $A^j$ ;

iii)  $(A^1, B^1)$  is called the *initial pair* of the chain, and the axiom  $A^1$  of the chain has the following property: no term ancestor (Def. 6.8) of  $A^1$  occurs either in a principal proposition (Def. 2.9)  $F$  of a Comp-rule occurrence in P or in a strong isolated descendant (Def. 6.13) of it;

iv) If  $(A^j, B^j), (A^{j+1}, B^{j+1})$  are consecutive pairs, one among the following conditions holds:

iv.1)  $B^j$  is a term ancestor of a sub-term of  $A^{j+1}$  through a term a.d. chain (Def. 6.8) in which no element different from  $B^j$  is a principal term (Def. 6.1) of a Comp-rule occurrence;

iv.2)  $B^j$  occurs as sub-term in  $A^{j+1}$ ;

iv.3)  $A^{j+1}$  occurs either in the principal proposition (Def. 2.9)  $F$  of the Comp-rule occurrence in which  $B^j$  is a principal term (Def. 6.1) or in a strong isolated descendant (Def. 6.13) of it;

v) The pair  $(A^m, B^m)$  is called the *end-pair* of the chain and  $B^m$  doesn't have any term descendant (Def. 6.8) which is sub-term of a maximal auxiliary term of a Comp-rule. Since  $B^m$  is a bound variable occurrence, it has an integral term descendant either in the root of P, or in a cut formula.

vi) If  $\mathcal{T}$  is a critical chain in P, a *critical sub-chain* of  $\mathcal{T}$  (briefly, a sub-chain of  $\mathcal{T}$ ) is any sub-sequence of  $\mathcal{T}$  which is a critical chain in P.

◇

**Remark 6.15.** In the definition above the crucial point is iv), and it must be noted that: in iv.1), iv.2) the link between  $B^j$  and  $A^{j+1}$  is standardly given by a term a.d. chain (Def. 6.8), while in iv.3) the link may be provided by an isolated a.d. chain (Def. 6.5) including strong propositional inferences.

◇

**Proposition 6.16.** *Definition 6.14 can be formally translated into the language of Primitive Recursive Arithmetic PRA, and then in that of Robinson's Arithmetic Q, so that the critical chain ADS's can be selected by a formal choice criterion (Def. 3.2) F and produce to the kind (Def. 3.3) Kr of the critical chains.*

*Proof:* An examination of Definition 6.14 shows that, given P, it describes some further recursive specific properties of the sequences  $\mathbf{R}$  and  $\mathcal{D}$ , which by hypothesis already satisfy the Comp ADS properties, that can be expressed by two  $\Delta_0$ -formulas  $G(\#P, \#\mathbf{R})$  and  $E(\#P, \#\mathbf{R}, \#\mathcal{D})$  as in the definition of choice criterion (Def. 3.2) is required. Thus, a formula  $F(\#P, x)$  having the form of a choice criterion can be constructed in the  $\mathbf{Q}$ -language as a translation of Definition 6.14. Moreover, since the points of Definition 6.14 can be easily transformed into the steps of a recursive procedure, if  $\exists y\Phi(\#P, \#\mathbf{R}, y)$  (see Def. 3.2) is true in the standard model of  $\mathbf{Q}$ , an effective procedure is definable that having P and  $\mathbf{R}$  as input gives as output a finite set  $\{\mathcal{D}_k\}$  of critical chains in P extracted from  $\mathbf{R}$ . Following Def. 3.3, we call *the kind Kr of the critical chain ADS's* the  $\mathbf{Q}$ -equivalence class of  $F(\#P, x)$ .

◇

**Example 6.17.** Let us consider the following proof H:

$$\frac{\frac{\frac{\frac{A_{o \rightarrow (o \rightarrow o)} B_o C_o \vdash A_{o \rightarrow (o \rightarrow o)} B_o C_o}{A_{o \rightarrow (o \rightarrow o)} B_o C_o \vdash \exists y_{o \rightarrow (o \rightarrow o)} [y_{o \rightarrow (o \rightarrow o)} B_o C_o]}{A_{o \rightarrow (o \rightarrow o)} B_o C_o \vdash \exists z_{o \rightarrow o} \exists y_{o \rightarrow (o \rightarrow o)} [z_{o \rightarrow o} C_o]}{\vdash A_{o \rightarrow (o \rightarrow o)} B_o C_o \supset \exists z_{o \rightarrow o} \exists y_{o \rightarrow (o \rightarrow o)} [z_{o \rightarrow o} C_o]} \supset\text{-R}}{\vdash \exists w_o [A_{o \rightarrow (o \rightarrow o)} B_o C_o \supset w_o]} \exists\text{-R}}{\neg \exists w_o [A_{o \rightarrow (o \rightarrow o)} B_o C_o \supset w_o] \vdash} \neg\text{-L}}{\forall x_o \neg \exists w_o [x_o \supset w_o] \vdash} \forall\text{-L}$$

The following  $\mathbf{C}$  is a **critical chain** in H:

$\mathbf{C}$ :  $\langle (A_{o \rightarrow (o \rightarrow o)}, y_{o \rightarrow (o \rightarrow o)}), (y_{o \rightarrow (o \rightarrow o)} B_o, z_{o \rightarrow o}), (\exists z_{o \rightarrow o} \exists y_{o \rightarrow (o \rightarrow o)} [z_{o \rightarrow o} C_o], w_o), (A_{o \rightarrow (o \rightarrow o)} B_o C_o, x_o) \rangle$   
the conditions of Definition 6.14 that provide the links between the pairs in  $\mathbf{C}$  are, respectively, iv.2), iv.1), iv.3);  
the strong propositional inference that contributes to  $\mathbf{C}$  is:  $\neg\text{-L}$ ;  $\mathbf{C}$ -axiom:  $A_{o \rightarrow (o \rightarrow o)}$ ;  $\mathbf{C}$ -theorem:  $A_{o \rightarrow (o \rightarrow o)} B_o C_o$ ;  
*Comp-measure*( $\mathbf{C}$ ) = 4; *inferential type*( $\mathbf{C}$ ) =  $(o \rightarrow (o \rightarrow o)) \rightarrow o$ . Observe that the weak propositional rule  $\supset\text{-R}$  does not break the critical chain, since the link between the second and the third pair of the chain is given by condition iv.1). ◇

**Example 6.18.** Consider the following proof P:

$$\frac{\frac{\frac{z_o \vdash z_o}{z_o \vdash \exists x_o(x_o)} \exists\text{-R} \quad A_{o \rightarrow o} y_o \vdash A_{o \rightarrow o} y_o}{A_{o \rightarrow o} y_o, z_o \vdash \exists x_o(x_o) \wedge A_{o \rightarrow o} y_o} \wedge\text{-R}}{A_{o \rightarrow o} y_o, z_o \vdash \exists u_o(\exists x_o(x_o) \wedge u_o)} \exists\text{-R}}{\frac{A_{o \rightarrow o} y_o, z_o \vdash \exists w_{(o \rightarrow o) \rightarrow o}(w_{(o \rightarrow o) \rightarrow o} \lambda u_o(\exists x_o(x_o) \wedge u_o))}{\neg \exists w_{(o \rightarrow o) \rightarrow o}(w_{(o \rightarrow o) \rightarrow o} \lambda u_o(\exists x_o(x_o) \wedge u_o)), A_{o \rightarrow o} y_o, z_o \vdash} \neg\text{-L}}{\forall h_o(h_o), A_{o \rightarrow o} y_o, z_o \vdash} \forall\text{-L}}{\frac{A_{o \rightarrow o} y_o, z_o \vdash \neg \forall h_o(h_o)}{A_{o \rightarrow o} y_o, z_o \vdash \exists f_{o \rightarrow o}(f_{o \rightarrow o} \forall h_o(h_o))} \exists\text{-R}}$$

The following  $\mathbf{C}_1$  and  $\mathbf{C}_2$  are **critical chains** in P:

$\mathbf{C}_1$ :  $\langle (z_o, x_o), (A_{o \rightarrow o} y_o, u_o), (\neg \exists w_{(o \rightarrow o) \rightarrow o} (w_{(o \rightarrow o) \rightarrow o} \lambda u_o (\exists x_o (x_o) \wedge u_o)), h_o), (\neg_{o \rightarrow o}, f_{o \rightarrow o}) \rangle$ ; the conditions of Definition 6.14 that provide the links between the pairs in  $\mathbf{C}_1$  are:  $\langle \text{iv.3}, \text{iv.3}, \text{iv.3} \rangle$   $\mathbf{C}_1$ -axiom:  $z_o$ ;  $\mathbf{C}_1$ -theorem:  $\neg_{o \rightarrow o}$ ;  $\text{Comp-measure}(\mathbf{C}_1) = 4$ ;  $\text{inferential type}(\mathbf{C}_1) = o \rightarrow (o \rightarrow o)$ ; observe that the third  $\exists$ -R downwards does not contribute to the chain; moreover, the strong propositional inferences that contribute to  $\mathbf{C}_1$  are:  $\wedge$ -R,  $\neg$ -L,  $\neg$ -R.  
 $\mathbf{C}_2$ :  $\langle (z_o, x_o), (A_{o \rightarrow o} y_o, u_o), (\exists_{(o \rightarrow o) \rightarrow o} w_{(o \rightarrow o) \rightarrow o}), (\neg \exists w_{(o \rightarrow o) \rightarrow o} (w_{(o \rightarrow o) \rightarrow o} \lambda u_o (\exists x_o (x_o) \wedge u_o)), h_o), (\neg_{o \rightarrow o}, f_{o \rightarrow o}) \rangle$ ; the conditions of Definition 6.14 that provide the links between the pairs in  $\mathbf{C}_1$  are:  $\langle \text{iv.3}, \text{iv.3}, \text{iv.3}, \text{iv.3} \rangle$ . Note that  $\mathbf{C}_1$  is a proper sub-chain of  $\mathbf{C}_2$ ; the two chains coincide up to the principal term  $u_o$  of the second  $\exists$ -R starting from the leftmost axiom downwards, from which  $\mathbf{C}_1$  goes directly to  $\neg \exists w_{(o \rightarrow o) \rightarrow o} (w_{(o \rightarrow o) \rightarrow o} \lambda u_o (\exists x_o (x_o) \wedge u_o))$ , while  $\mathbf{C}_2$  goes to the term  $\exists_{(o \rightarrow o) \rightarrow o}$ , occurring in the same principal proposition of  $u_o$ . Both the paths are allowed by the definition.  $\mathbf{C}_2$ -axiom:  $z_o$ ;  $\mathbf{C}_2$ -theorem:  $\neg_{o \rightarrow o}$ ;  $\text{Comp-measure}(\mathbf{C}_2) = 5$ ;  $\text{inferential type}(\mathbf{C}_2) = o \rightarrow (o \rightarrow o)$ .  
 $\diamond$

What is the essence of a critical chain? It is an ancestor-descendant relation among formulas of arbitrary types in a branch, which witnesses the action of a sequence of Comp-rules, with possible substantial intrusions of the principal proposition of a strong propositional rule. That is, propositional inference is mixed with the Comp-rule action, under the condition that it is completely extraneous with respect to the weakening. Without this mix, a chain constituted only by a term a.d. chain connecting auxiliary and principal terms of a sequence of Comp-rules distributed in a branch, would be much less expressive, and maybe trivial: it would not express the real impact of a Comp-rule occurrence, i.e. the formulas really included in its action field. Moreover, the inferential type of a critical chain is a fine expression of the action of  $\mathbf{LK}_\omega$  inference on types.

We shall see in the next sections that a complete semantics can be built on an abstraction operation on proofs (Def.3.9) that only saves segments determined by critical chains.

## 7. The inferential algebra $\text{Inf-}\mathcal{A}$ based on the critical chains

Having defined in Section 6 the ADS kind  $Kr$  of the critical chains, the task of this Section is to construct a class  $\text{Inf-}\mathcal{A}$  of inferential algebras based on  $Kr$  that provides the formal environment for the proof of Main Theorem 4.4. The definition of  $\text{Inf-}\mathcal{A}$  will be syntactic. Recalling Definition 3.9, first the domains of the algebra and the abstraction function (operation) $\Rightarrow$  (Section 7.1) are defined, then the application function (operation)  $*$  (Section 7.2) is introduced. Without any loss of generality, in the construction of the domains  $\{\text{KrD}_\alpha\}$  of  $\text{Inf-}\mathcal{A}$ , the end pair of the main critical chain of each tree occurs in an  $\exists$ -R Comp-rule. The alternative choice of a  $\forall$ -L Comp-rule would produce isomorphic inferential algebras. However, the preference for the rule  $\exists$ -R can be motivated by some developments that are shown in Section 10.

### 7.1. The domains $\{\text{KrD}_\alpha\}$ and the abstraction operation $\Rightarrow$ of $\text{Inf-}\mathcal{A}$

In the following we fix a canonical recursive bijection  $g_{(\alpha, \beta)} : \{A_\alpha : A_\alpha \text{ closed formula}\} \times \{A_\beta : A_\beta \text{ closed formula}\} \rightarrow \{A_{\alpha \rightarrow \beta} : A_{\alpha \rightarrow \beta} \text{ closed formula}\}$ <sup>9</sup>. However, our results do not depend on the choice of  $g$ , and may take  $\text{Inf-}\mathcal{A}$  to stand for the class of algebras obtained by varying  $g$ . Definition 7.1 presents the basis case domains of  $\text{Inf-}\mathcal{A}$ , i.e. those including only trees of inferential types  $o$  or  $i$ , and Lemma 7.2 constructively presents the abstraction operation  $\Rightarrow$  of  $\text{Inf-}\mathcal{A}$ , i.e it shows how the trees of higher  $Kr$ -inferential types are produced starting from the trees of elementary  $Kr$ -inferential types.<sup>10</sup>

**Definition 7.1.** The *inferential domains*  $\text{KrD}_o$  and  $\text{KrD}_i$  of the inferential algebra  $\text{Inf-}\mathcal{A}$  (Def. 3.9) based on the Comp-ADS kind  $Kr$ , are defined as follows:

i)  $\text{KrD}_o$  is the set of  $\mathbf{LK}_\omega$ -potential proof trees  $P$  such that:  $P$  has exactly one main critical chain  $\mathcal{T}$  of inferential type  $o$ ; the Comp-rule including the main critical chain is an  $\exists$ -R rule; the  $\mathcal{T}$ -theorem (coinciding with the  $\mathcal{T}$ -axiom, since  $\mathcal{T}$  has length 1) is any closed  $o$ -typed  $\mathbf{LK}_\omega$ -formula  $F_o$ ;  $F_o$  has all its uppermost term ancestors in  $P$

<sup>9</sup>An example is the following: let  $(\#C_{\alpha \rightarrow \beta}^j)_{j \in \mathbb{N}}$  be the sequence of the Gödel-numbers of the codomain formulas and  $(\#F_\alpha^r)_{r \in \mathbb{N}}, (\#G_\beta^m)_{m \in \mathbb{N}}$  be the sequences of the Gödel-numbers of the domain factor formulas, ordered by the relation  $<$  in  $\mathbb{N}$ ; let the sequence  $(\#F_\alpha^r, \#G_\beta^m)_{(r, m) \in \mathbb{N} \times \mathbb{N}}$  be ordered by the lexicographic order induced by the previous orders. Then, consider the orders induced in the corresponding set of formulas and define  $g_{(\alpha, \beta)}(\langle F_\alpha, G_\beta \rangle) \equiv$  'the element of the ordered codomain  $\{A_{\alpha \rightarrow \beta} : A_{\alpha \rightarrow \beta} \text{ closed formula}\}$  having the same place index of  $\langle F_\alpha, G_\beta \rangle$  in the ordered domain'.

<sup>10</sup>For a first idea of the construction, we suggest to concentrate on some cases presented at STEP 1 of the proof of Lemma 7.2 and, possibly, imagine the trees of  $\text{KrD}_i$  and  $\text{KrD}_o$  in the simplest forms, which are the ones explicitly indicated in Def. 8.1 for the sub-algebra  $\text{Inf-}\mathcal{P}$ . As to the definition of the application operation  $*$  of  $\text{Inf-}\mathcal{A}$ , it is essentially a corollary of the definition of the abstraction operation  $\Rightarrow$ : then, sub-section 7.2 can be read as a corollary of the proof of Lemma 7.2.

occurring in an axiom, and they must be integral term ancestors; all the introducing axioms must have the form:  $\Gamma \vdash F_o$ , with  $\Gamma$   $o$ -typed parameter (Def. 3.6) which has the Gödel-number of  $F_o$  as index.

ii)  $\text{KrD}_i$  is the set of  $\text{LK}_o$ -potential proof trees  $P$  such that:  $P$  has exactly one main critical chain  $\mathcal{T}$  of inferential type  $i$ ; the Comp-rule including the main critical chain is an  $\exists$ -R rule; the  $\mathcal{T}$ -theorem (coinciding with the  $\mathcal{T}$ -axiom, since  $\mathcal{T}$  has length 1) is any closed  $i$ -typed  $\text{LK}_o$ -formula  $A_i$ ;  $A_i$  has all uppermost term ancestors in  $P$  occurring in an axiom, and they must be integral term ancestors; all the introducing axioms must have the form:  $F_{i \rightarrow o} A_i \vdash F_{i \rightarrow o} A_i$ , where  $F_{i \rightarrow o}$  is the  $(i \rightarrow o)$ -typed non logical constant having as index the Gödel-number of  $A_i$ , and no term descendant of the left-side occurrence of  $A_i$  in  $P$  is an auxiliary term of any  $\lambda$ -rule.

◇

Note that the set of proofs  $\text{KrD}_i \cup \text{KrD}_o$  is large: it is already an expressive fragment of the  $\text{LK}_o$ -inference. The trees of  $\text{KrD}_i$  are simpler than the trees of  $\text{KrD}_o$  since  $h(i) = 0 < h(o) = 1$  and thus in any  $P \in \text{KrD}_o$  also critical chains of inferential type  $i$  may occur.

**Lemma 7.2.** (Main Lemma) Consider the trees of the domains  $\text{KrD}_o$  and  $\text{KrD}_i$  defined in Definition 7.1. Then, for each compound type  $\gamma \rightarrow \delta$  it is possible to construct an inferential domain  $\text{KrD}_{\gamma \rightarrow \delta}$  of Kr-modular trees of inferential type  $\gamma \rightarrow \delta$ , such that its elements are abstraction trees  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$ , with  $\mathbf{Q}^\gamma \in \text{KrD}_\gamma$ ,  $\mathbf{Q}^\delta \in \text{KrD}_\delta$ , having the following properties:

j)  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$  has exactly one main critical chain  $\mathcal{T}$  of inferential type  $\gamma \rightarrow \delta$ .

jj) The theorem  $G_\delta$  of  $\mathcal{T}$  has the form  $E_{\gamma \rightarrow \delta} C_\gamma$  with  $E_{\gamma \rightarrow \delta} \equiv g_{(\gamma, \delta)}(A_\gamma, B_\delta)$ , such that  $A_\gamma, B_\delta$  are the canonical components (Definition 3.9.iii) of the theorems of the main critical chains  $\mathcal{G}$  of  $\mathbf{Q}^\gamma$  and  $\mathcal{N}$  of  $\mathbf{Q}^\delta$ , and  $C_\gamma$  is the  $\gamma$ -typed syntactic parameter having as index the Gödel-number of the pair  $(\mathbf{Q}^\gamma, \mathbf{Q}^\delta)$ . The occurrence of  $C_\gamma$  in  $G_\delta$  is called the **memory parameter** of  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$ . The proof-segment containing  $\mathcal{T}$  and having the main Comp-rule as end rule is called the **main module** of  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$ .

jjj) Each uppermost term ancestor of the theorem  $G_\delta$  of  $\mathcal{T}$  occurs in a potential axiom  $\Gamma \vdash A$  or in a logical axiom  $A \vdash A$  of  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$ , and it must be an integral term ancestor. The same property holds for the memory parameter  $C_\gamma$  of  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$ .

*Proof of Main Lemma:* The construction of  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$  is done through the following steps:

**STEP 1:** First the **main module**  $\mathbf{P}^{\gamma \rightarrow \delta}$  of  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$  is constructed by induction on the complexity of types, and it is proven that  $\mathbf{P}^{\gamma \rightarrow \delta}$  already has the properties j)-jjj) stated by the thesis. The construction is such that the Comp-rule including the end pair of the main critical chain is an  $\exists$ -R rule.

Select the segment  $\mathbf{H}^\gamma$  of  $\mathbf{Q}^\gamma$ , (resp.  $\mathbf{H}^\delta$  of  $\mathbf{Q}^\delta$ ) having as end-rule the Comp-rule occurrence in which the theorem of the main critical chain  $\mathcal{G}$  (resp.  $\mathcal{N}$ ) occurs; in  $\mathbf{H}^\gamma$  (resp in  $\mathbf{H}^\delta$ ), consider the premise of the Comp-rule  $\exists$ -R having the theorem of the main critical chain  $\mathcal{G}$  (resp.  $\mathcal{N}$ ) as maximal auxiliary term; by construction this premise has the form:

$$X \vdash Y, F[\dots A_\gamma \dots] \text{ for } \mathbf{H}^\gamma \quad \text{and} \quad U \vdash V, N[\dots B_\delta \dots] \text{ for } \mathbf{H}^\delta$$

and the set of the axioms introducing the  $\mathcal{G}$ -theorem  $A_\gamma$  will be of the form either  $\{\Gamma^n \vdash A_\gamma : n = 1, \dots, t\}$  if  $\gamma$  is  $o$  or  $\{F_{\gamma \rightarrow o}^m A_\gamma \vdash F_{\gamma \rightarrow o}^m A_\gamma : m = 1, \dots, k\}$  if  $\gamma$  is  $i$ ; analogously for the  $\mathcal{N}$ -theorem  $B_\delta$ .

a) The segments  $\mathbf{H}^i, \mathbf{H}^o$  are the main modules  $\mathbf{P}^i, \mathbf{P}^o$  for the basis cases given by primitive types (Def. 7.1).

b) The main modules  $\mathbf{P}^{\gamma \rightarrow \delta}$  for the four compound type basis cases  $\gamma \rightarrow \delta \in \{o \rightarrow o, i \rightarrow o, o \rightarrow i, i \rightarrow i\}$  are the following:

$\mathbf{P}^{o \rightarrow o}$ , i.e.  $\gamma \equiv o, \delta \equiv o$ :

$$\frac{\frac{\frac{\frac{\Gamma^n \vdash A_o : n = 1, \dots, t}{\Pi_1} \quad X \vdash Y, F[\dots A_o \dots]}{X \vdash Y, F[\dots \mathbf{A}_o \dots]} \vee\text{-R}}{X \vdash Y, \exists x_o (F[\dots \mathbf{x}_o \dots] \vee h_o)} \exists\text{-R}}{X, U \vdash Y, V, (\exists x_o (F[\dots \mathbf{x}_o \dots] \vee h_o)) \wedge (N[\dots \mathbf{E}_{o \rightarrow o} \mathbf{y}_o \dots] \vee b_o)} \wedge\text{-R}}{\frac{X, U \vdash Y, V, \exists u_o ((\exists x_o (F[\dots \mathbf{x}_o \dots] \vee h_o)) \wedge (N[\dots \mathbf{u}_o \dots] \vee b_o))}{X, U \vdash Y, V, \exists u_o ((\exists x_o (F[\dots \mathbf{x}_o \dots] \vee h_o)) \wedge (N[\dots \mathbf{u}_o \dots] \vee b_o))} \exists\text{-R}}$$

where:  $\Pi_1$  is the proof-segment above the main Comp-rule in  $\mathbf{H}^\gamma$ . The proof-segment  $\Pi_2$  is obtained by replacing the closed  $B_o$  with  $E_{o \rightarrow o} y_o$  in each axiom  $\Gamma^r \vdash B_o, r = 1, \dots, k$ , in the proof-segment above the main Comp-rule

in  $\mathbf{H}^\delta$ ; through the hypotheses on  $\mathbf{Q}^\delta$ , which belongs to  $\text{Kr}\mathbf{D}_o$ , such replacement is possible.  $E_{o \rightarrow o}$  is the closed formula  $g_{(o,o)}(A_o, B_o)$  unambiguously determined by the recursive bijection  $g_{(o,o)}$ , and  $y_o$  is the  $o$ -typed syntactic parameter having as index the Gödel-number of the pair  $(\mathbf{Q}^\gamma, \mathbf{Q}^\delta)$ : it is the *memory parameter*.  $h_o$  and  $b_o$  are suitable non logical constants so that the index of the constant  $h_o$  is the Gödel-number of  $\mathbf{Q}^\gamma$ , the index of the constant  $b_o$  is the Gödel-number of  $\mathbf{Q}^\delta$ ; the two  $\vee$ -R-occurrences have been added in order to break possible new critical chain links of the form form iv.3)-Def.6.14 between some Comp-rule with maximal auxiliary term of type  $i$  in  $\Pi_1, \Pi_2$ , and the lower part of  $\mathbf{P}^{o \rightarrow o}$ . The *main critical chain* of  $\mathbf{P}^{o \rightarrow o}$  is:  $\mathcal{T} \equiv \langle (A_o, x_o), (E_{o \rightarrow o} y_o, u_o) \rangle$  with  $\mathcal{T}$ -axiom  $A_o$ ,  $\mathcal{T}$ -theorem  $E_{o \rightarrow o} y_o$  and inferential type  $o \rightarrow o$ ; the link between the pairs in  $\mathcal{T}$  is given by the second case of point iv.3) of Definition 6.14.

$\mathbf{P}^{i \rightarrow i}$ , i.e.  $\gamma \equiv i, \delta \equiv i$ :

$$\frac{\frac{\frac{\{F_{i \rightarrow o}^m A_i \vdash F_{i \rightarrow o}^m A_i : m = 1, \dots, t\}}{\Pi_1} \quad \frac{X \vdash Y, F[\dots \mathbf{A}_i \dots]}{X \vdash Y, \exists x_i F[\dots \mathbf{x}_i \dots]} \exists\text{-R}}{\frac{X, U \vdash Y, V, \exists x_i F[\dots x_i \dots] \wedge N[\dots \mathbf{E}_{i \rightarrow i} \mathbf{v}_i \dots]}{X, U \vdash \exists u_i (\exists x_i F[\dots x_i \dots] \wedge N[\dots \mathbf{u}_i \dots])} \exists\text{-R}}{\frac{\{F_{i \rightarrow o}^s (E_{i \rightarrow i} v_i) \vdash F_{i \rightarrow o}^s (E_{i \rightarrow i} v_i) : s = 1, \dots, k\}}{\Pi_2} \quad \frac{U \vdash V, N[\dots E_{i \rightarrow i} v_i \dots]}{U \vdash V, N[\dots E_{i \rightarrow o} w_i \dots]} \vee\text{-R}}{\frac{X, U \vdash Y, V, \exists x_i F[\dots x_i \dots] \wedge N[\dots \mathbf{E}_{i \rightarrow o} \mathbf{w}_i \dots] \vee d_o}{X, U \vdash Y, V, \exists u_o (\exists x_i F[\dots x_i \dots] \wedge N[\dots \mathbf{u}_o \dots] \vee d_o)} \exists\text{-R}} \wedge\text{-R}$$

where:  $\Pi_1$  is the proof-segment above the main Comp-rule in  $\mathbf{H}^\gamma$ . The proof-segment  $\Pi_2$  is obtained by replacing the closed  $B_i$  with  $E_{i \rightarrow i} v_i$  in each axiom  $F_{i \rightarrow o}^s B_i \vdash F_{i \rightarrow o}^s B_i, s = 1, \dots, k$ , in the proof-segment above the main Comp-rule in  $\mathbf{H}^\delta$ ; through the hypotheses on  $\mathbf{Q}^\delta$ , which belongs to  $\text{Kr}\mathbf{D}_i$ , such replacement is possible.  $E_{i \rightarrow i}$  is the closed formula  $g_{(i,i)}(A_i, B_i)$  unambiguously determined by the recursive bijection  $g_{(i,i)}$ , and  $v_i$  is the  $i$ -typed syntactic parameter having as index the Gödel-number of the pair  $(\mathbf{Q}^\gamma, \mathbf{Q}^\delta)$ : it is the *memory parameter*. Note that no Comp-rule may occur in the segments  $\Pi_1, \Pi_2$ , by definition of the trees of  $\text{Kr}\mathbf{D}_i$ . The *main critical chain* of  $\mathbf{P}^{i \rightarrow i}$  is:  $\mathcal{T} \equiv \langle (A_i, x_i), (E_{i \rightarrow i} v_i, u_i) \rangle$  with  $\mathcal{T}$ -axiom  $A_i$ ,  $\mathcal{T}$ -theorem  $E_{i \rightarrow i} v_i$  and inferential type  $i \rightarrow i$ ; the link between the pairs in  $\mathcal{T}$  is given by the second case of point iv.3) of Definition 6.14.

$\mathbf{P}^{i \rightarrow o}$ , i.e.  $\gamma \equiv i, \delta \equiv o$ :

$$\frac{\frac{\frac{\{F_{i \rightarrow o}^m A_i \vdash F_{i \rightarrow o}^m A_i : m = 1, \dots, t\}}{\Pi_1} \quad \frac{X \vdash Y, F[\dots \mathbf{A}_i \dots]}{X \vdash Y, \exists x_i F[\dots \mathbf{x}_i \dots]} \exists\text{-R}}{\frac{X, U \vdash Y, V, \exists x_i F[\dots x_i \dots] \wedge (N[\dots \mathbf{E}_{i \rightarrow o} \mathbf{w}_i \dots] \vee d_o)}{X, U \vdash Y, V, \exists u_o (\exists x_i F[\dots x_i \dots] \wedge (N[\dots \mathbf{u}_o \dots] \vee d_o))} \exists\text{-R}}{\frac{\{\Gamma^r \vdash E_{i \rightarrow o} w_i : r = 1, \dots, k\}}{\Pi_2} \quad \frac{U \vdash V, N[\dots E_{i \rightarrow o} w_i \dots]}{U \vdash V, N[\dots E_{i \rightarrow o} w_i \dots]} \vee\text{-R}}{\frac{X, U \vdash Y, V, \exists x_i F[\dots x_i \dots] \wedge (N[\dots \mathbf{E}_{i \rightarrow o} \mathbf{w}_i \dots] \vee d_o)}{X, U \vdash Y, V, \exists u_o (\exists x_i F[\dots x_i \dots] \wedge (N[\dots \mathbf{u}_o \dots] \vee d_o))} \exists\text{-R}} \wedge\text{-R}$$

where:  $\Pi_1$  is the proof-segment above the main Comp-rule in  $\mathbf{H}^\gamma$ . The proof-segment  $\Pi_2$  is obtained by replacing the closed  $B_o$  with  $E_{i \rightarrow o} w_i$  in each axiom  $\Gamma^r \vdash B_o, r = 1, \dots, k$ , in the proof-segment above the main Comp-rule in  $\mathbf{H}^\delta$ ; through the hypotheses on  $\mathbf{Q}^\delta$ , which belongs to  $\text{Kr}\mathbf{D}_o$ , such replacement is possible.  $E_{i \rightarrow o}$  is the closed formula  $g_{(i,o)}(A_i, B_o)$  unambiguously determined by the recursive bijection  $g_{(i,o)}$ , and  $w_i$  is the  $i$ -typed syntactic parameter having as index the Gödel-number of the pair  $(\mathbf{Q}^\gamma, \mathbf{Q}^\delta)$ : it is the *memory parameter*.  $d_o$  is a suitable non logical constant having as index the Gödel-number of  $\mathbf{Q}^\delta$ ; the  $\vee$ -R-occurrence has been added in order to break possible new critical chain links of the form iv.3)-Def.6.14 between some Comp-rule with maximal auxiliary term of type  $i$  in  $\Pi_2$  and the lower part of  $\mathbf{P}^{i \rightarrow o}$ . Note that no Comp-rule may occur in the segments  $\Pi_1$ , by definition of the trees of  $\text{Kr}\mathbf{D}_i$ . The *main critical chain* of  $\mathbf{P}^{i \rightarrow o}$  is:  $\mathcal{T} \equiv \langle (A_i, x_i), (E_{i \rightarrow o} w_i, u_o) \rangle$  with  $\mathcal{T}$ -axiom  $A_i$ ,  $\mathcal{T}$ -theorem  $E_{i \rightarrow o} w_i$  and inferential type  $i \rightarrow o$ ; the link between the pairs in  $\mathcal{T}$  is given by the second case of point iv.3) of Definition 6.14.

$\mathbf{P}^{o \rightarrow i}$ , i.e.  $\gamma \equiv o, \delta \equiv i$ :

$$\frac{\frac{\frac{\{\Gamma^n \vdash A_o : n = 1, \dots, t\}}{\Pi_1} \quad \frac{X \vdash Y, F[\dots \mathbf{A}_o \dots]}{X \vdash Y, F[\dots \mathbf{A}_o \dots]} \vee\text{-R}}{\frac{X \vdash Y, F[\dots \mathbf{A}_o \dots] \vee f_o}{X \vdash Y, \exists x_o F[\dots \mathbf{x}_o \dots] \vee f_o} \exists\text{-R}}{\frac{\{F_{i \rightarrow o}^m E_{o \rightarrow i} z_o \vdash F_{i \rightarrow o}^m E_{o \rightarrow i} z_o : m = 1, \dots, r\}}{\Pi_2} \quad \frac{U \vdash V, N[\dots E_{o \rightarrow i} z_o \dots]}{U \vdash V, N[\dots E_{o \rightarrow i} z_o \dots]} \vee\text{-R}}{\frac{X, U \vdash Y, V, (\exists x_o F[\dots x_o \dots] \vee f_o) \wedge N[\dots \mathbf{E}_{o \rightarrow i} \mathbf{z}_o \dots]}{X, U \vdash Y, V, \exists u_i ((\exists x_o F[\dots x_o \dots] \vee f_o) \wedge (N[\dots \mathbf{u}_i \dots]))} \exists\text{-R}} \wedge\text{-R}$$

where:  $\Pi_1$  is the proof-segment above the main Comp-rule in  $\mathbf{H}^\gamma$ . The proof-segment  $\Pi_2$  is obtained by replacing the closed  $B_i$  with  $E_{o \rightarrow i} z_o$  in each axiom  $F_{i \rightarrow o}^m B_i \vdash F_{i \rightarrow o}^m B_i, m = 1, \dots, r$ , in the proof-segment above the main Comp-rule in  $\mathbf{H}^\delta$ ; through the hypotheses on  $\mathbf{Q}^\delta$ , which belongs to  $\text{Kr}\mathbf{D}_i$ , such replacement is possible.  $E_{o \rightarrow i}$  is the closed

formula  $g_{(o,i)}(A_o, B_i)$  unambiguously determined by the recursive bijection  $g_{(o,i)}$ , and  $z_o$  is the o-typed syntactic parameter having as index the Gödel-number of the pair  $(Q^\gamma, Q^\delta)$ : it is the *memory parameter*.  $f_o$  is a suitable non logical constant having as index the Gödel-number of  $Q^\gamma$ ; the  $\vee$ -R-occurrence on the left has been added in order to break possible new critical chain links of the form iv.3)-Def.6.14 between some Comp-rule with maximal auxiliary term of type  $i$  in  $\Pi_1$  and the lower part of  $P^{o \rightarrow i}$ . Note that no Comp-rule may occur in the segments  $\Pi_2$ , by definition of the trees of  $\text{KrD}_i$ . The *main critical chain* of  $P^{o \rightarrow i}$  is  $\mathcal{T} \equiv \langle (A_o, x_o), (E_{o \rightarrow i} z_o, u_i) \rangle$  with  $\mathcal{T}$ -axiom  $A_o$ ,  $\mathcal{T}$ -theorem  $E_{o \rightarrow i} z_o$  and inferential type  $o \rightarrow i$ ; the link between the pairs in  $\mathcal{T}$  is given by the second case of point iv.3) of Definition 6.14.

Thus, properties j), jj), jjj) hold for the main module  $P^{\gamma \rightarrow \delta}$  of  $Q^\gamma \Rightarrow Q^\delta$  in the basis cases.

c) Construction of the main module  $P^{\gamma \rightarrow \delta}$  of  $Q^\gamma \Rightarrow Q^\delta$  at the compound-type induction step:

c1) Suppose that  $\delta \equiv \pi \rightarrow \eta$  and  $\gamma$  an arbitrary type. The construction of  $P^{\gamma \rightarrow \delta}$  will start from the main modules  $P^\gamma$  of  $Q^\gamma$  and  $P^\delta$  of  $Q^\delta$ . By induction hypothesis  $P^\gamma$  and  $P^\delta$  have properties j)-jjj) of the thesis. The following transformations will be performed on  $P^\gamma$ :

I) Consider the auxiliary proposition (Def. 2.9)  $A$  of the end-rule  $\exists$ -R in  $P^\gamma$ , that is the Comp-rule in which the end-pair of the main critical chain  $\mathcal{H}$  of  $P^\gamma$  occurs. By induction hypothesis, the  $\mathcal{H}$ -theorem, occurring in  $A$  as maximal auxiliary term of the main Comp-rule, has the form  $F_\gamma b_\alpha$  with either  $F_\gamma$  significant leftmost component (Def. 3.8) or  $b_\alpha$  empty if  $\gamma$  is a primitive type. Apply to the succedent of the  $P^\gamma$ -root a  $\vee$ -R having  $A$  as maximal non isolated introduced term (Def. 6.1.i). Let  $B$  be the principal proposition (Def. 2.9) of such  $\vee$ -R;

II) To the root of the obtained proof-segment, apply a  $\exists$ -R having  $B$  as auxiliary proposition and  $F_\gamma$  as maximal auxiliary term. In this way,  $F_\gamma$  becomes the axiom of a critical chain, since the  $\vee$ -R in which the uppermost  $F_\gamma$ -term ancestor occurs as non isolated principal term, is a weak propositional inference that does not produce any critical chain link.  $\exists B$  briefly indicates the principal proposition of the applied  $\exists$ -R and  $@P^\gamma$  the obtained potential proof. Consider now the proof  $P^\delta$ :

let  $\S P^\delta$  be the sub-proof of the premise of the lowermost  $\exists$ -R in  $P^\delta$ , that is the main Comp-rule, let  $C$  be the auxiliary proposition of such  $\exists$ -R. By induction hypothesis, the maximal auxiliary term of the rule, occurring in  $C$ , is the theorem of the main critical chain  $\mathcal{W}$  of  $P^\delta$  and has the form  $E_{\pi \rightarrow \eta} d_\pi$ , since  $\delta \equiv \pi \rightarrow \eta$ . The *canonical component*  $E_{\pi \rightarrow \eta}$  has been unambiguously determined at the construction of  $P^{\pi \rightarrow \eta}$  by the recursive bijection  $g_{(\pi, \eta)}$ , and has each its uppermost term ancestor in an axiom as an integral ancestor (Def. 6.8) which has the form  $E_{\pi \rightarrow \eta}$ . Moreover, by construction, such uppermost ancestor can be replaced by any term of the same type without breaking any rule constraint in  $P^\delta$ . The following transformations are performed on  $\S P^\delta$ :

J) uniformly replace in  $\S P^\delta$  the formula  $E_{\pi \rightarrow \eta}$  with  $G_{\gamma \rightarrow \delta} y_\gamma$ , being  $G_{\gamma \rightarrow \delta}$  the unique closed term  $\equiv g_{(\gamma, \delta)}(F_\gamma, E_{\pi \rightarrow \eta})$ , (recall that  $\delta \equiv \pi \rightarrow \eta$ ) and where  $y_\gamma$  is the  $\gamma$ -typed syntactic parameter having as index the Gödel-number of the pair  $(Q^\gamma, Q^\delta)$ , i.e. it is the *memory parameter*. Then, in the proposition  $C$ , the  $\mathcal{W}$ -theorem is also replaced by  $G_{\gamma \rightarrow \delta} y_\gamma d_\pi$ . Let  $C^*$  be the isolated formula produced by such transformation of  $C$ ;

JJ) to the so transformed premise of the lowermost  $\exists$ -R in  $P^\delta$ , apply a  $\vee$ -R having the non logical constant  $h_o$ , which has as index the Gödel-number of  $Q^\delta$ , as maximal non isolated introduced term (Def. 6.1.i). Let  $S$  be the obtained sequent. Do not apply any  $\exists$ -R to  $S$ . Let  $+\S P^\delta$  be the obtained potential proof of  $S$ .

Consider now the potential proofs  $@P^\gamma$  and  $+\S P^\delta$ . Apply to them the following rules:

JJJ) apply a  $\wedge$ -R between the root of  $@P^\gamma$  and the root of  $+\S P^\delta$ , having as principal proposition  $\exists B \wedge C^*$ ;

JV) then, apply to the conclusion of the  $\wedge$ -R above, a  $\exists$ -R having  $\exists B \wedge C^*$  as auxiliary proposition and  $G_{\gamma \rightarrow \delta} y_\gamma$  as maximal auxiliary term.

$P^{\gamma \rightarrow \delta}$  is the obtained tree.

The main critical chain of  $P^{\gamma \rightarrow \delta}$  is the one with axiom  $F_\gamma$  (see point II above) and theorem  $G_{\gamma \rightarrow \delta} y_\gamma$  (see point JV above), having inferential type  $\gamma \rightarrow \delta$ .  $G_{\gamma \rightarrow \delta}$  is the *canonical component* and  $y_\gamma$  the *memory parameter*. It is straightforward to see that points j)-jjj) of the thesis hold by construction for  $P^{\gamma \rightarrow \delta}$ . In particular, observe that the weak propositional rule  $\vee$ -R applied in I) and JJ) above breaks the production of any critical chain which could contradict the thesis. Moreover, the uniqueness of  $P^{\gamma \rightarrow \delta}$ , given the abstracted proofs, is in the fact that no ambiguity is in the choice of new parameters and constants, which are selected by the Gödel-numbers of the abstracted  $Q^\gamma, Q^\delta$ .

c2) The construction of  $P^{\gamma \rightarrow \delta}$  is strictly similar to c1) and b) above in the case where  $\gamma$  is a compound type  $\gamma \equiv \nu \rightarrow \sigma$  and  $\delta$  an arbitrary type, and the same conclusions hold.

**STEP 2:** If in both the abstracted  $Q^\gamma, Q^\delta$ , the end-rule is the main Comp-rule, then **stop:**  $Q^\gamma \Rightarrow Q^\delta$  coincides with the main module  $P^{\gamma \rightarrow \delta}$ .

**Otherwise,** construct the following potential proof tree  $\mathbf{R}^{(\gamma, \delta)}$ :

$$\frac{\frac{\frac{Q^\gamma}{\vdots}}{X \vdash Y} \quad \frac{\frac{Q^\delta}{\vdots}}{W \vdash \Omega}}{\frac{\vdash (\neg \bigwedge X) \vee \bigvee Y \quad \bigwedge W \wedge (\neg \bigvee \Omega) \vdash}{\vdash \neg((\neg \bigwedge X) \vee \bigvee Y \supset \bigwedge W \wedge (\neg \bigvee \Omega))} \supset\text{-L}}{\vdash \neg((\neg \bigwedge X) \vee \bigvee Y \supset \bigwedge W \wedge (\neg \bigvee \Omega))} \neg\text{-R}$$

where  $X \vdash Y$  is the root of  $\mathbf{Q}^\gamma$  and  $W \vdash \Omega$  is the root of  $\mathbf{Q}^\delta$ .  
Then  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$  is the following tree:

$$\frac{\frac{\mathbf{P}^{\gamma \rightarrow \delta}}{\vdots}}{U \vdash W, A} \quad \frac{\mathbf{R}^{(\gamma, \delta)}}{\vdots} \vdash C}{U \vdash W, A \wedge C} \wedge\text{-R}$$

where  $U \vdash W, A$  is the root of  $\mathbf{P}^{\gamma \rightarrow \delta}$ ,  $A$  is the principal proposition of the main Comp-rule, and  $\vdash C$  is the root of  $\mathbf{R}^{(\gamma, \delta)}$ .

Observe that, by construction, in  $\mathbf{R}^{(\gamma, \delta)}$  critical chains with inferential types  $\gamma$  and  $\delta$  necessarily occur, but no critical chain having inferential type with height  $h$  greater than or equal to  $h(\gamma \rightarrow \delta)$  may occur. Therefore, all points j)-jjj) of the thesis hold for the so constructed  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$ .  $\diamond$

**Remark 7.3.** The occurrences of  $o$ -typed constants, indexed by suitable Gödel-numbers, that are introduced by  $\vee$ -R rules through the construction of  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$  described above, are called *memory constants*. Both memory constants and memory parameters (Lemma 7.2, thesis jj)) can be replaced in the main module of  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$ , after a possible renaming of any quantified parameters in the uppermost segment of the module, with different parameters or constants of the same type, without breaking any rule constraint.  $\diamond$

**Definition 7.4.** For each compound type  $\gamma \rightarrow \delta$ , the *inferential domains*  $\text{Kr}\mathbf{D}_{\gamma \rightarrow \delta}$  and the abstraction operation  $\Rightarrow$  of the *inferential algebra*  $\text{Inf-}\mathcal{A}$  (Def. 3.9) based on the *Comp-ADS kind*  $Kr$ , are given by the  $Kr$ -modular trees of the form  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$  defined in the Main Lemma 7.2 starting from any pair  $\mathbf{Q}^\gamma, \mathbf{Q}^\delta$  of  $Kr$ -modular trees of the domains  $\text{Kr}\mathbf{D}_\gamma, \text{Kr}\mathbf{D}_\delta$ .  $\diamond$

**Proposition 7.5.** For each pair of types  $\gamma, \delta$ , the function  $\Rightarrow: \text{Kr}\mathbf{D}_\gamma \times \text{Kr}\mathbf{D}_\delta \rightarrow \text{Kr}\mathbf{D}_{\gamma \rightarrow \delta}$  of the inferential algebra  $\text{Inf-}\mathcal{A}$  is a recursive bijection.

*Proof:* By definition of  $\Rightarrow$  the image of the function coincides with the codomain; as to the injectivity, given an arbitrary  $\mathbf{Q}^{\gamma \rightarrow \delta} \equiv \mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$  in  $\text{Kr}\mathbf{D}_{\gamma \rightarrow \delta}$ , the abstracted proofs  $\mathbf{Q}^\gamma, \mathbf{Q}^\delta$  can be recursively reconstructed from the information included in the memory parameters occurring in the main module of  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$ ; such pair of trees is unique, since the memory parameters are indexed by their Gödel-numbers.

$\diamond$

**Remark 7.6.** The critical chains of the trees in the domains  $\{\text{Kr}\mathbf{D}_\alpha\}_{\alpha \in \text{types}}$  of  $\text{Inf-}\mathcal{A}$  have the following properties:  
i) For each type  $\gamma$ , each closed formula  $F_\gamma$  occurs in infinitely many different trees of the domains  $\{\text{Kr}\mathbf{D}_\alpha\}_{\alpha \in \text{types}}$  as the canonical component of the theorem of the main critical chain.  
ii) For each type  $\gamma$ , infinitely many different trees of the domains  $\{\text{Kr}\mathbf{D}_\alpha\}_{\alpha \in \text{types}}$  include Comp-rules having the maximal auxiliary term of type  $\gamma$ .

$\diamond$

## 7.2. The application operation $*$ of $\text{Inf-}\mathcal{A}$

At this point the application operation  $*$  of the inferential algebra must be introduced.

**Definition 7.7.** Recalling Definition 7.4, for each pair  $\gamma, \delta$  of types, the application function  $*$ :  $\text{Kr}\mathbf{D}_{\gamma \rightarrow \delta} \times \text{Kr}\mathbf{D}_\gamma \rightarrow \text{Kr}\mathbf{D}_\delta$  of the algebra  $\text{Inf-}\mathcal{A}$  (Def. 3.9) is defined as follows. Let  $\mathbf{Q}^{\gamma \rightarrow \delta} \in \text{Kr}\mathbf{D}_{\gamma \rightarrow \delta}$  and  $\mathbf{Q}^\gamma \in \text{Kr}\mathbf{D}_\gamma$ , and let  $F_{\gamma \rightarrow \delta}$  and  $B_\gamma$  be the canonical components of the theorems of the main critical chains resp. of  $\mathbf{Q}^{\gamma \rightarrow \delta}$  and of  $\mathbf{Q}^\gamma$ ; then  $\mathbf{Q}^{\gamma \rightarrow \delta} * \mathbf{Q}^\gamma$  is the potential proof-tree  $\mathbf{H}^\delta$  of  $\text{Kr}\mathbf{D}_\delta$  so chosen:

- i) consider  $\mathbf{Q}^{\gamma \rightarrow \delta}$  as  $\mathbf{R}^\gamma \Rightarrow \mathbf{R}^\delta$ ; by Proposition 7.5 both  $\mathbf{R}^\gamma, \mathbf{R}^\delta$  are recursively determined;
- ii) consider the main module  $\mathbf{P}^\delta$  of  $\mathbf{R}^\delta$ ;

ii.1) if the canonical component of  $\mathbf{P}^\delta$  is  $F_{\gamma \rightarrow \delta} B_\gamma$ , then  $\mathbf{Q}^{\gamma \rightarrow \delta} * \mathbf{Q}^\gamma$  is that element of  $\text{KrD}_\delta$  having the Gödel-number closest to the Gödel-number of  $\mathbf{P}^\delta$  among the Gödel-numbers of the trees of  $\text{KrD}_\delta$  that are obtained from  $\mathbf{P}^\delta$  by a replacement of memory constants and memory parameters;  
otherwise:

ii.2) in  $\mathbf{P}^\delta$ , the canonical component  $G_\delta$  is replaced with  $F_{\gamma \rightarrow \delta} B_\gamma$ , obtaining a tree  $\mathbf{M}^\delta$ . Consider the set  $M$  of the elements of  $\text{KrD}_\delta$  having  $F_{\gamma \rightarrow \delta} B_\gamma$  as the canonical component and that are identical to  $\mathbf{M}^\delta$  unless the indices of some non logical constants and syntactic parameters. Then, choose as  $\mathbf{Q}^{\gamma \rightarrow \delta} * \mathbf{Q}^\gamma$  the element of  $M$  having the Gödel-number closest to the Gödel-number of  $\mathbf{P}^\delta$ .

◇

It is provable that the function  $*$  is well defined and recursive.

**Lemma 7.8.** *Let  $\mathbf{P}^\delta$  be the main module of any arbitrary  $\mathbf{R}^\delta \in \text{KrD}_\delta$ . Then either  $\mathbf{P}^\delta$  is in  $\text{KrD}_\delta$  too, or a suitable replacement of memory parameters and memory constants is always possible in  $\mathbf{P}^\delta$ , so that the resulting tree  $\mathbf{H}$  is in  $\text{KrD}_\delta$ .*

For the proof see Appendix A2.

**Lemma 7.9.** *Let  $\mathbf{R}^\delta$  be any element of  $\text{KrD}_\delta$  that coincides with its main module, having an occurrence of the closed formula  $G_\delta$  as the canonical component; then for each closed  $\delta$ -typed formula  $F_\delta$  different from  $G_\delta$ , a tree  $\mathbf{H}^\delta$  in  $\text{KrD}_\delta$  exists, having  $F_\delta$  as the canonical component, so that  $\mathbf{H}^\delta$  is obtained from  $\mathbf{R}^\delta$  by uniformly replacing each term ancestor of the canonical component  $G_\delta$  by  $F_\delta$ , and by suitable replacements of non logical constants and memory parameters.*

For the proof see Appendix A3.

**Proposition 7.10.** *i) Given  $\mathbf{Q}^{\gamma \rightarrow \delta} \in \text{KrD}_{g \rightarrow \delta}$  and  $\mathbf{Q}^\gamma \in \text{KrD}_\gamma$  the function  $*$ :  $\text{KrD}_{\gamma \rightarrow \delta} \times \text{KrD}_\gamma \rightarrow \text{KrD}_\delta$  established by the previous definitions is well defined and recursive.*

*ii)  $\mathbf{H}^\delta \equiv \mathbf{Q}^{\gamma \rightarrow \delta} * \mathbf{Q}^\gamma$  can be obtained by suitable uniform replacements of terms and syntactic parameters from the main module of the right abstracted tree  $\mathbf{R}^\delta$  in  $\mathbf{R}^\gamma \Rightarrow \mathbf{R}^\delta \equiv \mathbf{Q}^{\gamma \rightarrow \delta}$ .*

*iii) The end-rule of  $\mathbf{H}^\delta \equiv \mathbf{Q}^{\gamma \rightarrow \delta} * \mathbf{Q}^\gamma$  is its main Comp-rule i.e.  $\mathbf{H}^\delta$  coincides with its main module.*

*Proof:*

i) The function  $*$  is well defined since, by Lemmas 7.8 and 7.9, the tree  $\mathbf{Q}^{\gamma \rightarrow \delta} * \mathbf{Q}^\gamma$  as defined in Definition 7.7 exists in  $\text{KrD}_\delta$ . As to the recursiveness, it can be noted that:  $k \equiv \text{Gödel number of } \mathbf{P}^\delta$  is computable; moreover, given a natural number it is possible to recursively establish if it is a Gödel number and, if so it is, what expression it codes. Then, starting from  $k$ , both downwards and upwards alongside the ordered natural numbers, it is possible to stop at the first Gödel number of a tree having the searched features that is found; if two numbers are found, the first above  $k$  and the second below  $k$ , the smallest is chosen. The process is effective and finite, since the searched tree exists. Moreover, taking into account the specific features of the employed Gödel numbering, a bound on the size of the Gödel number of the searched tree can be given starting from the Gödel numbers of the contracted trees.  
ii), iii) follow from Definition 7.7. ◇

**Corollary 7.11.** *For each potential proof  $\mathbf{Q}^{\gamma \rightarrow \delta} \in \text{KrD}_{\gamma \rightarrow \delta}$  the function  $\mathbf{Q}^{\gamma \rightarrow \delta} * (\cdot) : \text{KrD}_\gamma \rightarrow \text{KrD}_\delta$  is a recursive function.*

**Definition 7.12.** The class of the *inferential algebras of the critical chains* is  $\{\langle g, \{\text{KrD}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle\}$  where  $g$  varies in the set of the canonical recursive bijections, and the domains  $\text{KrD}_\alpha$  and the functions  $\Rightarrow, *$  are that described by Definitions 7.4, 7.7. ◇

It must be noted that the application function  $*$  of any inferential algebra  $\text{Inf-}\mathcal{A}$  is neither surjective nor injective. In fact, in  $\text{KrD}_{\gamma \rightarrow \delta}$  of  $\text{Inf-}\mathcal{A}$  infinitely many trees may exist, that have the same canonical component as the main ADS theorem, and that have the same main module unless the indices of some non logical constants and syntactic parameters. Then, fixing the types  $\gamma, \gamma \rightarrow \delta$ , a tree  $\mathbf{H}^\delta$  may be obtained as the application of infinitely many pairs of trees of the form  $(\mathbf{Q}^{\gamma \rightarrow \delta}, \mathbf{Q}^\gamma)$ . Also observe that infinitely many elements of  $\{\text{KrD}_\alpha\}_{\alpha \in \text{types}}$  cannot be the result of any application operation: in this class are all the trees having the canonical component which is not an application between terms.

## 8. Proof of the Main Theorem: the sub-algebra $Inf\text{-}\mathcal{P}$ of the inferential algebra $Inf\text{-}\mathcal{A}$

At the end of Section 7 it is remarked that the defined application operation  $*$  of  $Inf\text{-}\mathcal{A}$  is not injective. Thus, in this Section, a sub-algebra  $Inf\text{-}\mathcal{P}$  (Def 8.1) of  $Inf\text{-}\mathcal{A}$  is introduced, where  $*$  is injective and has much more regularity properties (Lemma 8.2). Then, an inferential interpretation  $\text{Kr}\mathbf{V}$ , sending typed formulas into elements of  $Inf\text{-}\mathcal{P}$ , can be defined (Def 8.4), where  $\text{Kr}\mathbf{V}$  is functionally sound (Theorem 8.5), so that all the ingredients to prove the main Theorem 4.4 are given (Corollary 8.6).

**Definition 8.1.** Given  $Inf\text{-}\mathcal{A} \equiv \langle g, \{\text{Kr}\mathbf{D}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$ , the sub-domains  $\{\text{Kr}\mathbf{W}_\alpha\}_{\alpha \in \text{types}}$ ,  $\text{Kr}\mathbf{W}_\alpha \subset \text{Kr}\mathbf{D}_\alpha$  for each  $\alpha$ , are so constructed:

i) the elements of  $\text{Kr}\mathbf{D}_o$  that belong to  $\text{Kr}\mathbf{W}_o$  have the following form:

$$\frac{\Gamma \vdash B_o}{\Gamma \vdash \exists x_o(x_o)} \exists\text{-R}$$

where  $\Gamma$  is the o-typed syntactic parameter having as index the Gödel-number of  $B_o$ ;

ii) the elements of  $\text{Kr}\mathbf{D}_i$  that belong to  $\text{Kr}\mathbf{W}_i$  have the following form:

$$\frac{F_{i \rightarrow o} A_i \vdash F_{i \rightarrow o} A_i}{F_{i \rightarrow o} A_i \vdash \exists x_i F_{i \rightarrow o} x_i} \exists\text{-R}$$

where  $F_{i \rightarrow o}$  is the  $i$ -typed non logical constant having as index the Gödel-number of  $A_i$ ;

iii) for each compound type  $\gamma \rightarrow \delta$  the sub-domain  $\text{Kr}\mathbf{W}_{\gamma \rightarrow \delta}$  is recursively constructed starting from  $\text{Kr}\mathbf{W}_i, \text{Kr}\mathbf{W}_o$  through the same procedures which have constructed the domains  $\{\text{Kr}\mathbf{D}_\alpha\}_{\alpha \in \text{types}}$  of  $Inf\text{-}\mathcal{A}$ , i.e. through the same abstraction operation  $\Rightarrow$ . We set  $Inf\text{-}\mathcal{P} \equiv \langle g, \{\text{Kr}\mathbf{W}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$

◇

*It must be noted that the sub-domain  $\text{Kr}\mathbf{W}_\alpha$  is very small with respect to the domain  $\text{Kr}\mathbf{D}_\alpha$ . To each tree  $\mathbf{P}$  in  $\text{Kr}\mathbf{W}_\alpha$  infinitely many trees correspond in  $\text{Kr}\mathbf{D}_\alpha - \text{Kr}\mathbf{W}_\alpha$  having  $\mathbf{P}$  as sub-proof.*

The following Lemma establishes that  $Inf\text{-}\mathcal{P} \equiv \langle g, \{\text{Kr}\mathbf{W}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$  is a sub-algebra of  $Inf\text{-}\mathcal{A}$  having relevant properties of trees in the sub-domains  $\{\text{Kr}\mathbf{W}_\alpha\}_{\alpha \in \text{types}}$ :

**Lemma 8.2.** *Let  $Inf\text{-}\mathcal{P} \equiv \langle g, \{\text{Kr}\mathbf{W}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$  where  $\{\text{Kr}\mathbf{W}_\alpha\}_{\alpha \in \text{types}}$  are the sub-domains defined in Definition 8.1. Then:*

- i) *Any element of  $\text{Kr}\mathbf{W}_{\gamma \rightarrow \delta}$  is the abstraction in  $Inf\text{-}\mathcal{A}$  of a unique pair of trees belonging resp. to  $\text{Kr}\mathbf{W}_\gamma$  and  $\text{Kr}\mathbf{W}_\delta$ .*
- ii) *For each type  $\gamma$ , for each closed formula  $F_\gamma$ , exactly one potential proof tree  $\mathbf{H}$  in  $\text{Kr}\mathbf{W}_\gamma$  exists so that  $F_\gamma$  is the canonical component of the theorem of the main critical chain of  $\mathbf{H}$ .*
- iii) *If a tree  $\mathbf{Q}$  belonging to  $\text{Kr}\mathbf{D}_\alpha$  coincides with its main module, and it is equal, unless the indices of some non logical constants and syntactic parameters, to an element  $\mathbf{P}$  of  $\text{Kr}\mathbf{W}_\alpha$  having the same canonical component, then  $\mathbf{Q}$  belongs to  $\text{Kr}\mathbf{W}_\alpha$  and must coincide with  $\mathbf{P}$ .*
- iv) *The elements of  $\text{Kr}\mathbf{W}_\alpha$  have the smallest height (see Section 2.4) among the main modules of the elements of  $\text{Kr}\mathbf{D}_\alpha$ .*
- v) *Let  $*$  be the application operation of the algebra  $Inf\text{-}\mathcal{A}$ ; then if  $\mathbf{P}^{\gamma \rightarrow \delta} \in \text{Kr}\mathbf{W}_{\gamma \rightarrow \delta}$  and  $\mathbf{P}^\gamma \in \text{Kr}\mathbf{W}_\gamma$ , the tree  $\mathbf{P}^{\gamma \rightarrow \delta} * \mathbf{P}^\gamma$  belongs to  $\text{Kr}\mathbf{W}_\delta$ . Therefore, taking into account the point i) too,  $Inf\text{-}\mathcal{P} \equiv \langle g, \{\text{Kr}\mathbf{W}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$  is really a sub-algebra of  $Inf\text{-}\mathcal{A}$ .*
- vi) *For each pair  $(\gamma, \delta)$  of types the function  $* : \text{Kr}\mathbf{W}_{\gamma \rightarrow \delta} \times \text{Kr}\mathbf{W}_\gamma \rightarrow \text{Kr}\mathbf{W}_\delta$  is injective.*

*Proof:*

i) By definition, each tree in  $\text{Kr}\mathbf{W}_{\gamma \rightarrow \delta}$  is obtained as an abstraction  $\mathbf{P}^{\gamma \rightarrow \delta} \Rightarrow \mathbf{P}^\delta$  with  $\mathbf{P}^\gamma \in \text{Kr}\mathbf{W}_\gamma, \mathbf{P}^\delta \in \text{Kr}\mathbf{W}_\delta$ , and such a pair must be unique by the properties of the function  $\Rightarrow$  of  $Inf\text{-}\mathcal{A}$  (Proposition 7.5).

ii) By induction on types:

the basis case is obvious by the definition of  $\text{Kr}\mathbf{W}_i$  and  $\text{Kr}\mathbf{W}_o$ : in these domains, the canonical component coincides with the main chain theorem, and different trees with the same theorem of the main critical chain are not possible. Induction step: suppose for a contradiction that in  $\text{Kr}\mathbf{W}_{\gamma \rightarrow \delta}$  two different trees  $\mathbf{H}_1$  and  $\mathbf{H}_2$  exist having the same canonical component  $F_{\gamma \rightarrow \delta}$ . Then, both  $\mathbf{H}_1$  and  $\mathbf{H}_2$  must be the abstraction of different pairs of trees, with the left abstracted one belonging to  $\text{Kr}\mathbf{W}_\gamma$  and the right abstracted one to  $\text{Kr}\mathbf{W}_\delta$ . Moreover, both in  $\mathbf{H}_1$  and in  $\mathbf{H}_2$ , by construction  $F_{\gamma \rightarrow \delta} \equiv g_{(\gamma, \delta)}(B_\gamma, E_\delta)$  with  $B_\gamma$  the canonical component of the left abstracted tree and  $E_\delta$  the canonical component of the right abstracted tree. But, by induction hypothesis, only one tree exists in  $\text{Kr}\mathbf{W}_\gamma$  having  $B_\gamma$  as canonical component, and only one tree exists in  $\text{Kr}\mathbf{W}_\delta$  having  $E_\delta$  as canonical component; this forces  $\mathbf{H}_1 \equiv \mathbf{H}_2$ .

iii) By induction on types:

it can be noted that both  $\mathbf{Q}$  and  $\mathbf{P}$  belong to  $\text{KrD}_\alpha$  and that  $\mathbf{Q}$  can be obtained from  $\mathbf{P}$  by suitable uniform replacements of non logical constants and syntactic parameters; moreover the canonical component is the same in both trees. Then, if  $\alpha \in \{i, o, o \rightarrow o, i \rightarrow o, o \rightarrow i, i \rightarrow i\}$  the thesis is self-evident, by construction both of  $\text{KrW}_\alpha$  and of  $\text{KrD}_\alpha$  (see STEP1.b of the proof of Main Lemma 7.2; note that in the hypotheses iii), the segments  $\Pi_1$  and  $\Pi_2$  of the trees there constructed become trivial). If  $\alpha$  is  $\gamma \rightarrow \delta$ , different from a basis case,  $\mathbf{Q}$  is an abstraction tree  $\mathbf{Q}^\gamma \Rightarrow \mathbf{Q}^\delta$  and the hypotheses of iii) must then hold for the abstracted trees too, by the definition of  $\Rightarrow$ . Then, by induction hypothesis,  $\mathbf{Q}^\gamma \in \text{KrW}_\gamma$ ,  $\mathbf{Q}^\delta \in \text{KrW}_\delta$  and  $\mathbf{Q}$  must belong to  $\text{KrW}_{\gamma \rightarrow \delta}$ .

iv) By induction on types: if  $\alpha \in \{i, o, o \rightarrow o, i \rightarrow o, o \rightarrow i, i \rightarrow i\}$  the thesis is self-evident, since all trees are minimal in the respective domains. The induction step is quite natural by construction of the domains: if the abstracted trees have the minimal height in the respective domains, so it must be for the abstraction tree.

v) Let  $F_{\gamma \rightarrow \delta}$  be the canonical component of  $\mathbf{P}^{\gamma \rightarrow \delta}$  and  $B_\gamma$  be the canonical component of  $\mathbf{P}^\gamma$ ; by Definition 7.7 of  $*$ , the canonical component of  $\mathbf{P}^{\gamma \rightarrow \delta} * \mathbf{P}^\gamma$  is  $F_{\gamma \rightarrow \delta} B_\gamma$ . By definition of  $*$  and by construction of the sub-domains  $\text{KrW}_{\gamma \rightarrow \delta}$  and  $\text{KrW}_\gamma$  to which the contracted trees  $\mathbf{P}^{\gamma \rightarrow \delta}$  and  $\mathbf{P}^\gamma$  belong, the application  $\mathbf{P}^{\gamma \rightarrow \delta} * \mathbf{P}^\gamma$  may differ from the unique element  $\mathbf{H}$  of  $\text{KrW}_\delta$  having  $F_{\gamma \rightarrow \delta} B_\gamma$  as canonical component, only for the indices of some non logical constants and syntactic parameters. Therefore, by iii) above,  $\mathbf{P}^{\gamma \rightarrow \delta} * \mathbf{P}^\gamma$  is in  $\text{KrW}_{\gamma \rightarrow \delta}$  and so it must coincide with  $\mathbf{H}$ .

vi) The canonical component of an application tree  $\mathbf{P}$  must be a term application of the form  $F_{\gamma \rightarrow \delta} B_\gamma$ , and the contracted trees must have as canonical components respectively  $F_{\gamma \rightarrow \delta}$  and  $B_\gamma$ . Therefore, by point ii) above, at most one pair of contracted trees belonging to  $\{\text{KrW}_\alpha\}_{\alpha \in \text{types}}$  and producing  $\mathbf{P}$  may exist.

◇

*Remark to point vi) above:* Due to definition (7.7) of  $*$ , the thesis 8.2.vi) does not imply that  $\mathbf{P}$  cannot be produced through application by any pair of trees which is not in  $\{\text{KrW}_\alpha\}_{\alpha \in \text{types}}$ : indeed, having fixed the resulting application tree and the left contracted tree, many trees in  $\text{KrD}_\gamma$  may be the right contracted tree .

**Proposition 8.3.** *The trees of the sub-algebra  $\text{Inf-}\mathcal{P}$  are uniform proofs in the sense of Miller et al. [23].*

*Proof:* By construction, from Definition 8.1 and Lemma 8.2.

◇

On the contrary, in general, the trees of the whole  $\text{Inf-}\mathcal{A}$  are not uniform proofs.

**Definition 8.4.** Given an inferential algebra  $\text{Inf-}\mathcal{A}$  based on the ADS kind  $Kr$ , the corresponding *interpretation of  $\text{LK}_\omega$  based on the ADS kind  $Kr$*  is the function

$$\text{KrV} : \bigcup_{\alpha \in \text{types}} \{B_\alpha : B_\alpha \text{ closed formula of type } \alpha\} \rightarrow \text{Inf-}\mathcal{A}$$

such that  $\text{KrV}(F_\delta)$  is the unique potential proof-tree of the sub-algebra  $\text{Inf-}\mathcal{P}$  having  $F_\delta$  as the canonical component of the theorem of the main critical chain. ◇

As an example, in the Appendix A1, the  $\text{KrV}$ -interpretation in a fixed  $\text{Inf-}\mathcal{A}$  of the universal quantification  $\forall_{(o \rightarrow o) \rightarrow o}$  is given.

Observe that infinitely many interpretations of the form  $\text{KrV}$  are possible, since by varying the canonical bijection  $g$  infinitely many different  $Kr$ -based inferential algebras  $\text{Inf-}\mathcal{A}$  can be constructed. The difference is not merely technical, since e.g., with a fixed  $B_\alpha$  different  $g$ 's  $g_1$  and  $g_2$  may give rise to trees  $\text{KrV}_1(B_\alpha)$  and  $\text{KrV}_2(B_\alpha)$  having very different complexities resp. in the algebras  $\text{Inf-}\mathcal{A}_1$  and  $\text{Inf-}\mathcal{A}_2$ .

**Theorem 8.5.** *The interpretation  $\text{KrV}$  defined in Def. 8.4 is functionally sound (Def. 4.2).*

*Proof:*  $\text{KrV}(A_{\gamma \rightarrow \delta}) * \text{KrV}(B_\gamma)$  is identical to  $\text{KrV}(A_{\gamma \rightarrow \delta} B_\gamma)$ : indeed, by properties of  $\text{Inf-}\mathcal{P}$ , the tree on the left side, resulting from the  $*$ -application, must belong to  $\text{Inf-}\mathcal{P}$  (closure property w.r.t.  $*$ ) and, by definition of  $*$ , it must have  $A_{\gamma \rightarrow \delta} B_\gamma$  as the canonical component; then, it must be the unique tree of  $\text{Inf-}\mathcal{P}$  having such property. For the same reason,  $\text{KrV}(A_\gamma) \Rightarrow \text{KrV}(B_\delta)$  is identical to  $\text{KrV}(g(A_\gamma, B_\delta))$  ◇

**Corollary 8.6.** *The thesis of Main Theorem 4.4 holds for the kind  $Kr$  .*

*Proof:* It is straightforward to see that the inferential algebra  $\text{Inf-}\mathcal{A}$  based on the ADS kind  $Kr$  of the critical chains, the sub-algebra  $\text{Inf-}\mathcal{P}$  of  $\text{Inf-}\mathcal{A}$ , and the interpretation  $\text{KrV}$  of closed formulas into  $\text{Inf-}\mathcal{P}$  defined above, satisfy the thesis of the Theorem. ◇

## 9. $\mathbf{LK}_\omega$ -Completeness with respect to inferential models

In this Section it is shown that if  $\text{Inf-}\mathcal{A} \equiv \langle g, \{\text{KrD}_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$  is the inferential algebra based on the ADS kind  $Kr$  of the critical chains, then infinitely many frames (Def. 5.2) in the class  $\langle (\{\text{KrD}_\alpha\}, \text{KrV}), \Lambda, \cong \rangle$  are sound functional denotations w.r.t  $\mathbf{LK}_\omega$  (Def. 5.3). The result is essentially based on Theorem 8.5 that establishes that  $\text{KrV}$  is functionally sound. As a consequence, the thesis of the Inferential Semantics Theorem 5.9 can be proven, that is  $\mathbf{LK}_\omega$  is sound and complete with respect to a sub-class of  $\langle (\{\text{KrD}_\alpha\}, \text{KrV}), \Lambda, \cong \rangle$ . The following technical notions must be mentioned:

**Definition 9.1.** Let  $\Lambda$  be a consistent extension of  $\mathbf{LK}_\omega$  such that the  $\Lambda$ -deduction apparatus has exactly the same logical axioms and logical rules as  $\mathbf{LK}_\omega$  and the  $\Lambda$ -language is an expansion of the  $\mathbf{LK}_\omega$ -language through at most a denumerable set of new primitive non logical constants for each type  $\gamma$ . Then we say that  $\Lambda$  is a *definitive maximal extension* of  $\mathbf{LK}_\omega$  if the following conditions hold:

- i)  $\Lambda$  is maximal, i.e. for each  $o$ -typed formula  $A$  of the  $\Lambda$ -language either  $\Lambda \vdash A$  or  $\Lambda \vdash \neg A$ ;
- ii) for each  $o$ -typed formula  $A$  of  $\Lambda$  having  $a_\alpha$  as only free variable

$$\Lambda \vdash \neg \forall x_\alpha A \supset \neg [c_\alpha / a_\alpha] A$$

where  $c_\alpha$  is a suitable primitive non logical constant added to the language of  $\mathbf{LK}_\omega$ <sup>11</sup>

◇

**Lemma 9.2.** 1) If  $\Delta$  is a consistent maximal extension of  $\mathbf{LK}_\omega$  the following hold: i) if  $A, B$  are  $o$ -typed formulas we have that  $\Delta \vdash A \supset B$  if and only if either  $\Delta \not\vdash A$  or  $\Delta \vdash B$ ; ii) if  $\forall x_\alpha A$  is closed and  $\Delta \vdash \forall x_\alpha A$ , then  $\Delta \vdash [h_\alpha / a_\alpha] A$  for each closed  $\alpha$ -typed formula  $h_\alpha$ ,  $a_\alpha$  free variable corresponding to  $x_\alpha$ .  
2) For each consistent extension  $\Delta$  of  $\mathbf{LK}_\omega$  having the same language, a definitive maximal extension  $\Lambda$  of  $\Delta$  exists.

The non-straightforward part of 9.2 is the point 2); the proof is classical and has different presentations; see [3, 26, 5].

**Theorem 9.3.** Let  $\mathcal{M}_\Lambda \equiv \langle (\{\text{KrD}_\alpha\}, \text{KrV}), \Lambda, \cong \rangle$  be an inferential frame based on the ADS kind  $Kr$  and the algebra  $\text{Inf-}\mathcal{A}$ , where  $\Lambda$  is a definitive maximal extension of  $\mathbf{LK}_\omega$ . Then  $\mathcal{M}_\Lambda$  is a sound functional denotation (Def. 5.3) for the logical constants and the equality symbols  $=_\alpha$  of the  $\mathbf{LK}_\omega$ -language.

*Proof:* It must be noted that, since  $\Lambda$  is a definitive maximal  $\mathbf{LK}_\omega$ -extension, for each  $o$ -typed  $A$  either  $\text{KrV}(A) \cong \text{KrV}(\top)$  or  $\text{KrV}(A) \cong \text{KrV}(\perp)$  holds. It is sufficient to show that i), iii), v), vii) of Definition 5.3 are satisfied.

a)  $[\text{KrV}(\bigwedge_{o \rightarrow (o \rightarrow o)} B_o)] * \text{KrV}(C_o) \cong \text{KrV}(\top)$  implies, by Th. 8.5,  $\text{KrV}(\bigwedge_{o \rightarrow (o \rightarrow o)} B_o) * \text{KrV}(C_o) \cong_\Lambda \text{KrV}(\top)$  and then  $\text{KrV}(\bigwedge_{o \rightarrow (o \rightarrow o)} B_o) C_o \cong \text{KrV}(\top)$  which, by definition of  $\cong$  in an inferential frame, implies

$\Lambda \vdash (\bigwedge_{o \rightarrow (o \rightarrow o)} B_o) C_o =_o \top$ ; thus, by properties of  $\Lambda$ , it follows  $\Lambda \vdash B_o$  and  $\Lambda \vdash C_o$ , which are equivalent to  $\Lambda \vdash B_o =_o \top$  and to  $\Lambda \vdash C_o =_o \top$ , and so, by definition of  $\cong$ , both  $\text{KrV}(B_o) \cong \text{KrV}(\top)$  and  $\text{KrV}(C_o) \cong \text{KrV}(\top)$  hold. On the other side,  $\text{KrV}(B_o) \cong \text{KrV}(\top)$  and  $\text{KrV}(C_o) \cong \text{KrV}(\top)$ , by properties of  $\Lambda$ , imply  $\Lambda \vdash (\bigwedge_{o \rightarrow (o \rightarrow o)} B_o) C_o =_o \top$  which gives, by definition of  $\cong$ ,  $\text{KrV}(\bigwedge_{o \rightarrow (o \rightarrow o)} B_o) C_o \cong \text{KrV}(\top)$ , from which, through Th. 8.5,  $[\text{KrV}(\bigwedge_{o \rightarrow (o \rightarrow o)} B_o)] * \text{KrV}(C_o) \cong \text{KrV}(\top)$ .

b)  $\text{KrV}(\neg_{o \rightarrow o} A_o) * \text{KrV}(A_o) \cong \text{KrV}(\top)$  implies, by Th. 8.5,  $\text{KrV}(\neg_{o \rightarrow o} A_o) \cong \text{KrV}(\top)$  which, by definition of  $\cong$  gives  $\Lambda \vdash \neg_{o \rightarrow o} A_o =_o \top$  that, by properties of  $\Lambda$ , implies  $\Lambda \vdash A_o =_o \perp$  which, by definition of  $\cong$ , gives  $\text{KrV}(A_o) \cong \text{KrV}(\perp)$ ; on the other side,  $\text{KrV}(A_o) \cong \text{KrV}(\perp)$ , by definition of  $\cong$  and by properties of  $\Lambda$ , gives  $\Lambda \vdash \neg_{o \rightarrow o} A_o =_o \top$  from which  $\text{KrV}(\neg_{o \rightarrow o} A_o) \cong \text{KrV}(\top)$ , that, by Th. 8.5, is  $\text{KrV}(\neg_{o \rightarrow o}) * \text{KrV}(A_o) \cong \text{KrV}(\top)$ .

c) From  $\text{KrV}(\bigvee_{(a \rightarrow o) \rightarrow o} \lambda x_\alpha B_o) * \text{KrV}(B_o) \cong \text{KrV}(\top)$ , by Th. 8.5, we have that  $\text{KrV}(\bigvee_{(a \rightarrow o) \rightarrow o} \lambda x_\alpha B_o) \cong \text{KrV}(\top)$  which, by definition of  $\cong$ , gives  $\Lambda \vdash \bigvee_{(a \rightarrow o) \rightarrow o} \lambda x_\alpha B_o$ . Since  $\Lambda$  is a definitive maximal  $\mathbf{LK}_\omega$ -extension (Definition 9.1, Lemma 9.2), for each closed  $h_\alpha$  of the  $\Lambda$ -language  $\Lambda \vdash [h_\alpha / a_\alpha] B_o$  holds,  $a_\alpha$  free variable corresponding to  $x_\alpha$ , and, since  $\Lambda$  must include the  $\lambda$ -rule, it proves  $[h_\alpha / a_\alpha] B_o =_o (\lambda x_\alpha B_o) h_\alpha$ . Then  $\text{KrV}((\lambda x_\alpha B_o) h_\alpha) \cong \text{KrV}(\top)$  holds for each closed  $h_\alpha$ , and so, by Th. 8.5,  $\text{KrV}(\lambda x_\alpha B_o) * \text{KrV}(h_\alpha) \cong \text{KrV}(\top)$  for each closed  $h_\alpha$  is obtained. On the other

<sup>11</sup>Note that ii) is often called Henkin condition, see [26] p 45, [5] p 119. Moreover, the maximality condition i) can be written equivalently: for each  $o$ -typed  $A$  of the  $\Lambda$ -language either  $\Lambda \vdash A =_o \top$  or  $\Lambda \vdash \neg A =_o \top$ .

side,  $\text{KrV}(\lambda x_\alpha B_o) * \text{KrV}(h_\alpha) \cong \text{KrV}(\top)$  for each closed formula  $h_\alpha$  of the  $\Lambda$ -language, gives  $\text{KrV}((\lambda x_\alpha B_o)h_\alpha) \cong \text{KrV}(\top)$  and then, by definition of  $\cong$  and by the  $\lambda$ -rule of  $\Lambda$ , gives  $\Lambda \vdash [h_\alpha/a_\alpha]B_o$  for each closed formula  $h_\alpha$ . It must hold  $\Lambda \vdash \bigvee_{(\alpha \rightarrow o) \rightarrow o} \lambda x_\alpha B_o$ : otherwise, by Definition 9.1.i) it should be  $\Lambda \vdash \neg \bigvee_{(\alpha \rightarrow o) \rightarrow o} \lambda x_\alpha B_o$ , that, by Lemma 9.2 ii) forces  $\Lambda \vdash \neg [c_\alpha/a_\alpha]B_o$  for some closed term  $c_\alpha$  of the  $\Lambda$ -language, which is a contradiction. Therefore,  $\text{KrV}(\bigvee_{(\alpha \rightarrow o) \rightarrow o} \lambda x_\alpha B_o) \cong \text{KrV}(\top)$ , that, by Th. 8.5, gives  $\text{KrV}(\bigvee_{(\alpha \rightarrow o) \rightarrow o} * \text{KrV}(\lambda x_\alpha B_o)) \cong \text{KrV}(\top)$ .

**d)** If  $[\text{KrV}(=\_{\alpha \rightarrow (\alpha \rightarrow o)}) * \text{KrV}(A_\alpha)] * \text{KrV}(B_\alpha) \cong \text{KrV}(\top)$  then, by Th. 8.5,  $\text{KrV}((=\_{\alpha \rightarrow (\alpha \rightarrow o)}) A_\alpha) B_\alpha \cong \text{KrV}(\top)$  which, by definition of  $\cong$  and the usual abbreviations, gives  $\Lambda \vdash A_\alpha =_\alpha B_\alpha$ , that is the definition of  $\text{KrV}(A_\alpha) \cong \text{KrV}(B_\alpha)$ . On the other side  $\text{KrV}(A_\alpha) \cong \text{KrV}(B_\alpha)$  by definition gives  $\Lambda \vdash A_\alpha =_\alpha B_\alpha$  which, by definition of  $\cong$ , gives  $\text{KrV}((=\_{\alpha \rightarrow (\alpha \rightarrow o)}) A_\alpha) B_\alpha \cong \text{KrV}(\top)$  that, by Theorem 8.5, gives  $[\text{KrV}(=\_{\alpha \rightarrow (\alpha \rightarrow o)}) * \text{KrV}(A_\alpha)] * \text{KrV}(B_\alpha) \cong \text{KrV}(\top)$ .  $\diamond$

**Theorem 9.4.** *Let  $\mathcal{M}_\Lambda \equiv \langle (\{\text{KrD}_\alpha\}, \text{KrV}), \Lambda, \cong \rangle$  be an inferential frame based on  $Kr$  and  $\text{Inf-}\mathcal{A}$ , where  $\Lambda$  is a definitive maximal extension of  $\mathbf{LK}_\omega$ . Then  $\mathcal{M}_\Lambda$  is an inferential structure based on  $\{\text{KrD}_\alpha\}$ .*

*Proof:* Theorem 9.3. implies that  $\mathcal{M}_\Lambda$  is a sound denotation; it must be proven that the condition of Definition 5.6 holds. It can be noted that  $\text{KrV}_\varphi(\lambda x_\alpha A_\beta)$  is  $\text{KrV}((\lambda x_\alpha A_\beta)^\varphi) \in D_{\alpha \rightarrow \beta}$  by definition of  $\text{KrV}_\varphi$ . Consider an arbitrary  $\text{KrV}(F_\alpha) \in D_\beta, F_\alpha$  closed formula; then by Theorem 8.5  $\text{KrV}((\lambda x_\alpha A_\beta)^\varphi) * \text{KrV}(F_\alpha)$  coincides with  $\text{KrV}((\lambda x_\alpha A_\beta)^\varphi F_\alpha)$  and, through the  $\lambda$ -rule of  $\Lambda$ ,  $\text{KrV}((\lambda x_\alpha A_\beta)^\varphi F_\alpha) \cong \text{KrV}(A_\beta^\varphi[F_\alpha/b_\alpha])$  is obtained, where  $b_\alpha$  is the free variable corresponding to  $x_\alpha$ , that by definition is different from any free variable occurring in  $\lambda x_\alpha A_\beta$ . But, by construction, if  $\psi$  is defined as the  $b_\alpha$ -variant of  $\varphi$  given by  $\psi(b_\alpha) = \text{KrV}(F_\alpha)$ , then  $\text{KrV}(A_\beta^\varphi[F_\alpha/b_\alpha])$  is  $\text{KrV}(A_\beta^\psi)$ ; thus,  $\text{KrV}((\lambda x_\alpha A_\beta)^\varphi) * \text{KrV}(F_\alpha) \cong \text{KrV}_\psi(A_\beta)$  holds, from which  $\text{KrV}_\varphi(\lambda x_\alpha A_\beta) * \text{KrV}(F_\alpha) \cong \text{KrV}_\psi(A_\beta)$  follows.  $\diamond$

**Corollary 9.5.** *(Soundness) Let  $A$  be a sentence in the language of  $\mathbf{LK}_\omega$ ; then  $\mathbf{LK}_\omega \vdash A$  implies that  $A$  is true in each inferential structure  $\mathcal{M}_\Lambda \equiv \langle (\{\text{KrD}_\alpha\}, \text{KrV}), \Lambda, \cong \rangle$ .*

*Proof:*  $\mathbf{LK}_\omega \vdash A$  implies  $\Lambda \vdash A$  for each extension of  $\mathbf{LK}_\omega$ , that is  $\Lambda \vdash A =_o \top$ , which is  $\text{KrV}(A) \cong \text{KrV}(\top)$ .  $\diamond$

Recall that the notion of truth is meaningful only if  $\mathcal{M}_\Lambda$  is an inferential structure and not simply an inferential frame. That is, the fundamental condition is that  $\mathcal{M}_\Lambda$  is a sound functional denotation; if this is missing, the hypothesis  $\mathbf{LK}_\omega \vdash A$  would not be sufficient for the truth of  $A$  in  $\mathcal{M}_\Lambda$ .

**Corollary 9.6.** *(Existence of an inferential model) Let  $\mathcal{G}$  be a consistent set of  $o$ -typed closed formulas in the language of  $\mathbf{LK}_\omega$ ; then  $\mathcal{G}$  has an inferential model.*

*Proof:* By Lemma 9.2,  $\Delta = \mathcal{G} \cup \mathbf{LK}_\omega$  admits a maximal definitive extension  $\Lambda$ . By Theorem 9.4, if the ADS kind  $Kr$  of the critical chains is chosen,  $\mathcal{M}_\Lambda \equiv \langle (\{\text{KrD}_\alpha\}, \text{KrV}), \Lambda, \cong \rangle$  is an inferential structure, and, moreover  $\Lambda \vdash B =_o \top$  holds for each  $B$  in  $\mathcal{G}$ . Then  $\mathcal{M}_\Lambda$  is an inferential model for  $\mathcal{G}$ .  $\diamond$

**Theorem 9.7.** *(Completeness of  $\mathbf{LK}_\omega$  w.r.t. the inferential semantics) If the sentence  $A$  is true in each inferential structure of the form  $\langle (\{\text{KrD}_\alpha\}, \text{KrV}), \Lambda, \cong \rangle$ , then it is  $\mathbf{LK}_\omega$ -provable.*

*Proof:* By hypothesis  $\text{KrV}(A) \cong \text{KrV}(\top)$  for each  $\Lambda$  definitive maximal extension of  $\mathbf{LK}_\omega$ . Assume for a contradiction that  $\mathbf{LK}_\omega \not\vdash A$ ; then  $\mathbf{LK}_\omega \cup \{\neg A\}$  is consistent and let  $\Sigma$  be a definitive maximal extension of it. It follows that  $A$  must be true in  $\mathcal{M}_\Sigma$ , that is  $\text{KrV}(A) \cong_\Sigma \text{KrV}(\top)$ , which is  $\Sigma \vdash A =_o \top$  i.e.  $\Sigma \vdash A$ , which is a contradiction, since by construction  $\Sigma \vdash \neg A$  and  $\Sigma$  is consistent.  $\diamond$

The results presented in this Section give the proof of the Inferential Semantics Theorem 5.9.

## 10. Meaning and semantical complexity

The inferential interpretation  $\text{KrV}(B_\alpha)$  in the domain  $\text{KrD}_\alpha$  in an inferential structure

$\mathcal{M}_\Lambda \equiv \langle (\{\text{KrD}_\alpha\}, \text{KrV}), \Lambda, \cong \rangle$  based on the ADS-kind  $Kr$  of the critical chains is a constructive semantical object which suitably expresses both the logical structure and the typed structure of  $B_\alpha$ . Only after the construction of the interpretation tree  $\text{KrV}(B_\alpha)$ , a notion of semantical identification between sentences in the structure is given, of the form  $\text{KrV}(A_\alpha) \cong \text{KrV}(B_\alpha)$ , that for the  $o$ -typed sentences provides a denotation by truth or falsehood, through the identifications  $\text{KrV}(A) \cong \text{KrV}(\top)$  or  $\text{KrV}(A) \cong \text{KrV}(\perp)$ . As it has been shown, by the Inferential Semantics Theorem 5.9, the interpretation  $\text{KrV}$  and such notion of truth have the soundness and completeness properties. But it is also clear that the truth denotation, notwithstanding these important properties, is poor with respect to the richness of information included in the interpretation-tree  $\text{KrV}(B_\alpha)$ ; moreover it does not supply well defined semantics features for formulas.

It is therefore interesting to introduce a notion of *meaning* of a closed formula  $B_\alpha$  in an arbitrary inferential frame  $\langle (\{\text{K}D_\alpha\}, \text{V}), \Delta, \cong \rangle$ : it is a sequence of objects recursively extracted from the tree  $\text{V}(B_\alpha)$ , that is *starting from the semantic level*. It will be essentially defined as the set of the Comp-Axiom instances corresponding to the Comp-rules which produces the Comp-ADS's in the tree  $\text{V}(B_\alpha)$ , and furthermore to those Comp-rules occurring in the interpretations  $\text{V}(F_\gamma)$ 's of the elements  $F_\gamma$ 's of the Comp-ADS's occurring in  $\text{V}(B_\alpha)$ , and so on. In the case of the inferential models based on the ADS kind  $Kr$  of the critical chains, due to the peculiar structure of the trees  $\text{KrV}(B_\alpha)$ , the process ends and gives a finite sequence of Comp-Axiom instances that we can consider as constructing  $B_\alpha$  from the semantical point of view. That is, the Comp-Axiom produces the semantics of the interpreted higher order sentence, and a suitable finite sequence of Comp-Axioms, univocally determined by the interpretation  $\text{KrV}$ , is the meaning of the sentence. Moreover, a notion of composition between meanings can be naturally defined, and it can be applied in the  $Kr$  based frames.

A question arises: why can't the tree  $\text{V}(B_\alpha)$  itself be an acceptable notion of meaning of  $B_\alpha$ ? The answer is that the interpretation  $\text{V}(B_\alpha)$  potentially includes the meaning (by its critical chains and its inferential types), but does not express it declaratively. The aim of the work developed in this section is to define a meaning function that maps terms to a sequence of declarative statements, that is to say, of properties of the term. Moreover, such definition of meaning is particularly manageable: it can be extended to the meaning of a rule in a proof  $Q$ , of a critical chain in  $Q$ , and to the meaning of the proof  $Q$ . This can also extend to an inferential semantics of proofs, the basic outline of which is sketched in Section 11. It must be noted that, anyway, the proposed definition of meaning is one among many similar possible notions, and that many different *abstraction procedures* acting on  $\text{V}(B_\alpha)$  could be done, in addition to that here defined, if more specific theoretical or applicative purposes are considered. The following is both simple and expressive.

**Definition 10.1.** Let  $\langle (\{\text{K}D_\alpha\}, \text{V}), \Delta, \cong \rangle$  be an arbitrary inferential semantics frame for  $\mathbf{LK}_\omega$ . Let  $B_\alpha$  be a closed formula of the language. Then  $\text{Meaning}(B_\alpha)$  is a sequence of Comp-Axioms defined through the following *meaning abstraction procedure*:

I) Step 0: Let  $\mathbf{P}(B_\alpha)$  be the tree in  $\text{K}D_\alpha$  assigned by  $\text{V}$  to  $B_\alpha$ . Consider the set of the maximal Comp-ADS's of kind  $K$  in  $\mathbf{P}(B_\alpha)$ , also including the uppermost  $K$ -ADS's of Comp-measure 1 in  $\mathbf{P}(B_\alpha)$ , ordered in the following way: the tree branches are ordered from the leftmost to the rightmost, and in a same branch the upper ADS precedes the lower ADS; assign 1 to the uppermost ADS in the leftmost branch and proceed without counting any ADS twofold. Let  $\mathcal{T}_1, \dots, \mathcal{T}_m$  be the obtained ADS sequence. For each  $\mathcal{T}_j$  write the sub-sequence  $\mathcal{S}_{0j}$  of the terms occurring as Comp-rule auxiliary terms, obtaining the sequences  $\mathcal{S}_{01}, \dots, \mathcal{S}_{0m}$ ; then, write the sequence  $\mathcal{S}_0$  given by the concatenation of the  $\mathcal{S}_{0j}$ 's.

II) Step  $n + 1$ : If  $\mathcal{S}_n$  is not empty, let  $F_{n_1}, \dots, F_{n_{k_n}}$  be the formulas of  $\mathcal{S}_n$ . For each  $F_{n_i}$  consider the syntactic parameters possibly occurring in them as closed terms, construct the inferential interpretation tree  $\mathbf{P}(F_{n_i})$ , and work on it as at the step 0, obtaining a concatenation sequence  $\mathcal{S}_{(n+1)_i}$ ; if each  $\mathcal{S}_{(n+1)_i}$  is empty, stop; otherwise, write the sequence  $\mathcal{S}_{(n+1)}$  given by the concatenation of the  $\mathcal{S}_{(n+1)_i}$ 's, having deleted in each  $\mathcal{S}_{(n+1)_i}$  the occurrences of  $F_{n_i}$ .

III)  $\text{Meaning}(B_\alpha)$  is a sequence of Comp-Axioms such that each axiom has as auxiliary formula (see Section 3) the corresponding element of the sequence  $\Sigma$  obtained by the concatenation of the sequences  $\mathcal{S}_0, \dots, \mathcal{S}_n$ , and such that the principal variable has the same type as the auxiliary formula. In writing the mentioned Comp-Axioms instances, the syntactic parameters possibly occurring in the formulas of  $\Sigma$ , are considered as free variables.  $\text{Meaning}(B_\alpha)$  is also called the *explicit meaning* of  $B_\alpha$ . The *reduced meaning* of  $B_\alpha$ , which we also write  $\text{meaning}(B_\alpha)$ , is given by the sequence  $\Sigma$  whose element are the auxiliary formulas of the Comp Axioms occurring in  $\text{Meaning}(B_\alpha)$ .

◇

In general,  $\text{Meaning}(B_\alpha)$  may be infinite. Indeed, if  $K$  is an arbitrary ADS-kind on which an arbitrary inferential frame is based, it may happen, in any Comp-ADS  $\mathcal{T}$  occurring in a tree  $\mathbf{P}$  of the inferential algebra domains, that an

element  $C_\gamma$  is such that  $h(\gamma) > \text{inferential type}(\mathcal{T})$ . This implies that some interpretation trees considered through the meaning abstraction procedure may have strictly increasing inferential type heights, and then Comp-ADS's of strictly increasing complexity. Conversely, it will be proven that, if in  $\{\mathbb{K}D_\alpha\}$   $K \equiv Kr$  i.e. the algebra is based on the Comp-ADS  $Kr$  of the critical chain, the meaning is always finite.

**Example 10.2.** Here is given the computation of  $\text{Meaning}(\bigvee_{(o \rightarrow o) \rightarrow o})$  starting from the interpretation  $\text{KrV}(\bigvee_{(o \rightarrow o) \rightarrow o})$  which is the following tree (see also the Appendix A1):

$$\frac{\frac{\frac{\Gamma \vdash C_o}{\Gamma \vdash C_o \vee h_o} \vee\text{-R}}{\Gamma \vdash \exists x_o(x_o \vee h_o)} \exists\text{-R} \quad \frac{\Delta \vdash F_{o \rightarrow o} y_o}{\Delta \vdash F_{o \rightarrow o} y_o \vee b_o} \vee\text{-R}}{\Gamma, \Delta \vdash \exists x_o(x_o \vee h_o) \wedge (F_{o \rightarrow o} y_o \vee b_o)} \wedge\text{-R}}{\Gamma, \Delta \vdash \exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o))} \exists\text{-R}}{\frac{\Gamma, \Delta \vdash \exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (F_{o \rightarrow o} y_o \vee b_o))}{\Gamma, \Delta \vdash \exists q_{o \rightarrow o}(\exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (q_{o \rightarrow o} y_o \vee b_o)))} \vee\text{-R}}{\Gamma, \Delta \vdash \exists q_{o \rightarrow o}(\exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (q_{o \rightarrow o} y_o \vee b_o)))} \exists\text{-R}}$$

(consider the root of the proof segment above as the left premise of the  $\wedge\text{-R}$  rule below)

$$\frac{\frac{\Omega \vdash \bigvee_{(o \rightarrow o) \rightarrow o} \theta_{o \rightarrow o}}{\Omega \vdash \bigvee_{(o \rightarrow o) \rightarrow o} \theta_{o \rightarrow o} \vee d_o} \vee\text{-R}}{\Gamma, \Delta, \Omega \vdash \exists q_{o \rightarrow o}(\exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (q_{o \rightarrow o} y_o \vee b_o))) \wedge (\bigvee_{(o \rightarrow o) \rightarrow o} \theta_{o \rightarrow o} \vee d_o)} \wedge\text{-R}}{\Gamma, \Delta, \Omega \vdash \exists r_o(\exists q_{o \rightarrow o}(\exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (q_{o \rightarrow o} y_o \vee b_o))) \wedge (r_o \vee d_o))} \exists\text{-R}}$$

At the Step 0 of the meaning abstraction procedure, the sequence of the critical chains is:

$$\mathcal{T}_1 \equiv \langle \langle C_o, x_o \rangle \rangle$$

$$\mathcal{T}_2 \equiv \langle \langle C_o, x_o \rangle, \langle F_{o \rightarrow o} y_o, u_o \rangle \rangle$$

$$\mathcal{T}_3 \equiv \langle \langle F_{o \rightarrow o}, q_{o \rightarrow o} \rangle, \langle \bigvee_{(o \rightarrow o) \rightarrow o} \theta_{o \rightarrow o}, r_o \rangle \rangle$$

the sequences  $\mathcal{S}_0, \dots, \mathcal{S}_3$  are:

$$\mathcal{S}_0 \equiv \langle C_o \rangle$$

$$\mathcal{S}_1 \equiv \langle C_o, F_{o \rightarrow o} y_o \rangle$$

$$\mathcal{S}_2 \equiv \langle F_{o \rightarrow o}, \bigvee_{(o \rightarrow o) \rightarrow o} \theta_{o \rightarrow o} \rangle$$

The concatenation sequence  $\mathcal{S}_0$  is:  $\langle C_o, C_o, F_{o \rightarrow o} y_o, F_{o \rightarrow o}, \bigvee_{(o \rightarrow o) \rightarrow o} \theta_{o \rightarrow o} \rangle$

At the Step 1 of the meaning abstraction procedure, the interpretation trees of each element of  $\mathcal{S}_0$  must be considered, and for each tree the sequence of the critical chains must be written. Such trees either are already presented in the Appendix, or are straightforwardly computable since are referred to  $o$ -typed formulas. So the sequences  $\mathcal{S}_1$ , can be directly written:

$$\mathcal{S}_{1_1} \equiv \langle C_o \rangle$$

$$\mathcal{S}_{1_2} \equiv \langle C_o \rangle$$

$$\mathcal{S}_{1_3} \equiv \langle F_{o \rightarrow o} y_o \rangle$$

$$\mathcal{S}_{1_4} \equiv \langle C_o, F_{o \rightarrow o} y_o \rangle$$

$$\mathcal{S}_{1_5} \equiv \langle \bigvee_{(o \rightarrow o) \rightarrow o} \theta_{o \rightarrow o} \rangle$$

For each  $\mathcal{S}_{1_i}$ , the procedure prescribes to delete each occurrence of the formula which is interpreted by the tree corresponding to  $\mathcal{S}_{1_i}$ , and then the concatenation sequence  $\mathcal{S}_1$  is:  $\langle C_o, F_{o \rightarrow o} y_o \rangle$ .

At the Step 2 we have:

$$\mathcal{S}_{2_1} \equiv \langle C_o \rangle$$

$$\mathcal{S}_{2_2} \equiv \langle F_{o \rightarrow o} y_o \rangle$$

and then  $\mathcal{S}_2$  is empty.

The final concatenation sequence  $\Sigma$  is  $\mathcal{S}_0 \mathcal{S}_1$ , that is :

$\langle C_o, C_o, F_{o \rightarrow o} y_o, F_{o \rightarrow o}, \bigvee_{(o \rightarrow o) \rightarrow o} \theta_{o \rightarrow o}, C_o, F_{o \rightarrow o} y_o \rangle$ . The explicit meaning  $\text{Meaning}(\bigvee_{(o \rightarrow o) \rightarrow o})$  in the given inferential frame is:

$$\langle \exists x_o(x_o \leftrightarrow C_o), \exists x_o(x_o \leftrightarrow C_o), \forall y_o \exists x_o(x_o \leftrightarrow F_{o \rightarrow o} y_o), \exists x_{o \rightarrow o}(x_{o \rightarrow o} =_{o \rightarrow o} F_{o \rightarrow o}), \forall \theta_{o \rightarrow o} \exists x_o(x_o \leftrightarrow \bigvee_{(o \rightarrow o) \rightarrow o} \theta_{o \rightarrow o}), \exists x_o(x_o \leftrightarrow C_o), \forall y_o \exists x_o(x_o \leftrightarrow F_{o \rightarrow o} y_o) \rangle$$

Note that, as to the definition of semantical complexity measures, the order of the axioms in the sequence is relevant, and that several occurrences of the same axioms contribute to the peculiarity of the meaning. We can so describe the declarative content of the meaning of  $\bigvee_{(o \rightarrow o) \rightarrow o}$  as expressed by the sequence  $\Sigma$  above: some formula

of type  $o$  exists, and some functional  $F$  exists (the 4th element of the sequence) and this  $F$  is total (defined on all  $o$ -typed formulas) and  $\forall$  is total on all such functionals.  $\diamond$

It must be noted that the meaning of  $B_\alpha$  depends only on  $(\{\text{K}D_\alpha\}, \mathbf{V})$ , i.e. on the inferential interpretation, and that it does not depend on the equivalence class of semantical values assigned by the structure to  $B_\alpha$ . It is the comparison between meanings that will depend on the whole frame. Furthermore, the notion of meaning does not privilege the type  $o$ , as happens for the notion of truth: each typed sentence has a meaning in an inferential semantics frame, that expresses its semantical complexity and that is recursively computable.

**Definition 10.3.** Let  $\mathcal{M} \equiv \langle (\{\text{K}D_\alpha\}, \mathbf{V}), \Delta, \cong \rangle$  be an arbitrary inferential semantics frame for  $\mathbf{LK}_\omega$ . Let  $B_\alpha, C_\beta$  be closed formulas, and let  $\text{meaning}(B_\alpha), \text{meaning}(C_\beta)$  be their reduced meanings (Def. 10.1). Then :

i)  $B_\alpha, C_\beta$  have *isomorphic meanings* in  $\mathcal{M}$  if a bijection  $\mu: \{\text{elements of meaning}(B_\alpha)\} \rightarrow \{\text{elements of meaning}(C_\beta)\}$  exists, such that: the meaning sequence order, is preserved i.e.  $\mu(E_j) \equiv G_j$ , with  $j$  the meaning sequence index, and the type is preserved, i.e.  $\text{type}(\mu(A)) \equiv \text{type}(A)$  for each  $E_j, G_j, A$  in the respective meaning sequences  $\text{meaning}(B_\alpha), \text{meaning}(C_\beta)$ .

ii)  $B_\alpha, C_\beta$  have *the same meaning* in  $\mathcal{M}$  if they have isomorphic meanings in the frame by a meaning bijection  $\mu: \{\text{elements of meaning}(B_\alpha)\} \rightarrow \{\text{elements of meaning}(C_\beta)\}$  and, moreover,  $\mathbf{V}(B_\alpha) \cong \mathbf{V}(C_\beta)$  and  $\mathbf{V}(A) \cong \mathbf{V}(\mu(A))$  for each  $A$  occurring in  $\text{meaning}(B_\alpha)$ .

$\diamond$

Therefore, if  $B_\alpha, C_\beta$  have the same meaning in a given inferential structure, then they must have in it the same truth value, i.e. they belong to the same the semantical equivalence class established by  $\cong$ ; in general, the converse does not hold.

**Proposition 10.4.** Let  $\mathcal{M} \equiv \langle (\{\text{K}rD_\alpha\}, \text{K}r\mathbf{V}), \Delta, \cong \rangle$  be an inferential semantics frame for  $\mathbf{LK}_\omega$  based on the ADS kind  $Kr$ . Then for each  $o$ -typed sentence  $B_o$ ,  $\text{Meaning}(B_o)$  is  $\langle A \rangle$  where  $A$  is the universal closure of  $\exists x_o(x_o \longleftrightarrow B_o)$ , and for each  $i$ -typed formula  $C_i$ ,  $\text{Meaning}(C_i)$  is  $\langle E \rangle$  where  $E$  is the universal closure of  $\exists x_i(x_i =_i C_i)$ .

*Proof:* It is straightforward from the definition of  $(\{\text{K}rD_\alpha\}, \text{K}r\mathbf{V})$  and from the Definition 10.1 of meaning.  $\diamond$

Note that the meaning of an  $o$ -typed sentence  $B$  is essentially an existential statement referred to  $B$ : in a sense, an ontological statement of the existence of  $B$ .

**Definition 10.5.** Let  $\mathcal{M} \equiv \langle (\{\text{K}D_\alpha\}, \mathbf{V}), \Delta, \cong \rangle$  be an arbitrary inferential semantics frame for  $\mathbf{LK}_\omega$ . Then, the composition  $\odot$  between meanings in the frame is given as follows:  $\text{Meaning}(B_{\alpha \rightarrow \beta}) \odot \text{Meaning}(C_\alpha)$  is the output of the meaning abstraction procedure (Definition 10.1) applied to the tree  $\mathbf{V}(B_{\alpha \rightarrow \beta}) * \mathbf{V}(C_\alpha)$ . In the frame the meaning is *compositional*, i.e. the composition  $\odot$  is well defined, if, for each pair  $(B_{\alpha \rightarrow \beta}, C_\alpha)$ :

$$\text{Meaning}(B_{\alpha \rightarrow \beta}) \odot \text{Meaning}(C_\alpha) \text{ coincides with } \text{Meaning}(B_{\alpha \rightarrow \beta} C_\alpha)$$

and the three meaning sequences written above, either all are infinite or all are finite.  $\diamond$

The following surprising property of the inferential algebras of the form  $\text{Inf-}\mathcal{A}$  is crucial for the compositionality of the meaning:

**Lemma 10.6.** Let  $P$  be a potential proof tree of a  $Kr$ -based inferential algebra  $\text{Inf-}\mathcal{A} \equiv \langle \{\text{K}rD_\alpha\}_{\alpha \in \text{types}}, \Rightarrow, * \rangle$ ; then the main *Comp-ADS* occurring in  $P$ , i.e. the main critical chain  $\mathcal{T}$  of  $P$ , has a length  $\leq 2$ .

*Proof:* the thesis follows from the construction of the domains  $\text{K}rD_\alpha$ , presented in the proof of Lemma 7.2 in Section 7.  $\diamond$

**Theorem 10.7.** In the class of the inferential semantics structures  $\langle (\{\text{K}rD_\alpha\}, \text{K}r\mathbf{V}), \Lambda, \cong \rangle$  based on the ADS-kind  $Kr$  of the critical chains, the meaning of any closed formula  $B_\alpha$  is finite and compositional.

*Proof:*

a) the meaning is finite: by induction on the height of the type  $\alpha$  in  $\text{Meaning}(B_\alpha)$ : if  $\alpha$  is a primitive type the thesis follows from Proposition 10.4. Let  $\alpha$  be the compound type  $\alpha \rightarrow \beta$ : the *Comp-ADS*'s that must be considered in the meaning abstraction procedure are the critical chains (Definition 6.14) of the tree  $\text{K}r\mathbf{V}(B_\alpha)$ .  $\text{K}r\mathbf{V}(B_\alpha)$  is a monic

tree with a main critical chain of inferential type  $\alpha$ , and a well known distribution of non-main critical chains with their inferential types. By Lemma 10.6 the main critical chain  $\mathcal{T}$  of the tree is formed by two pairs of formulas, and so only the types of the  $\mathcal{T}$ -axiom and of the  $\mathcal{T}$ -theorem, that produce the inferential type of the tree, occur in  $\mathcal{T}$ . Then, the types of the maximal auxiliary terms of the Comp-rules of the main chain must have an height strictly smaller than  $h(\alpha)$ , and so it is for all the remaining critical chains of the tree. Then, in the trees  $\text{KrV}(F_\gamma)$ 's produced from  $\text{KrV}(B_\alpha)$  by the further meaning abstraction procedure steps, the inferential type heights of the critical chains strictly decrease, that is  $h(\gamma) < h(\alpha)$  for each  $F_\gamma$ . When inferential type heights  $\leq 1$  are reached, they correspond to primitive types, and the procedure necessarily stops.

b) The meaning is compositional. Having proved that the meanings are always finite, the thesis is a consequence of Theorem 8.5. Indeed, it must hold:

$$\text{KrV}((B_{\alpha \rightarrow \beta})) * \text{KrV}((C_\alpha)) \equiv \text{KrV}((B_{\alpha \rightarrow \beta} C_\alpha))$$

then, by definition of the meaning composition  $\odot$  and of meaning :

$$\text{Meaning}(B_{\alpha \rightarrow \beta}) \odot \text{Meaning}(C_\alpha) \equiv \text{Meaning}(B_{\alpha \rightarrow \beta} C_\alpha)$$

where the composition is well defined, since the meaning sequences are finite.  $\diamond$

**Example 10.8.** Let  $B_{i \rightarrow o}$  and  $C_i$  be closed formulas, and let  $\langle (\{\text{KrD}_\alpha\}, \text{KrV}), \Delta, \cong \rangle$  be an inferential semantics frame.  $\text{Meaning}(C_i)$  is  $\langle \exists x_i (C_i =_i x_i) \rangle$  (Prop. 10.4). Let us compute  $\text{Meaning}(B_{i \rightarrow o})$ . The tree  $\text{KrV}(B_{i \rightarrow o})$  is the abstraction tree  $\text{KrV}(N_i) \Rightarrow \text{KrV}(E_o)$  with  $(N_i, E_o) \equiv g_{(i,o)}^{-1}(B_{i \rightarrow o})$ ,  $g$  canonical bijection of the algebra,  $N_i, E_o$  closed formulas. The following are the trees  $\text{KrV}(N_i)$  and  $\text{KrV}(E_o)$  (Def. 8.1, Def. 8.4):

$$\frac{F_{i \rightarrow o} N_i \vdash F_{i \rightarrow o} N_i}{F_{i \rightarrow o} N_i \vdash \exists x_i F_{i \rightarrow o} x_i} \exists\text{-R} \quad \frac{\Delta \vdash E_o}{\Delta \vdash \exists z_o (z_o)} \exists\text{-R}$$

where  $F_{i \rightarrow o}$  is the  $i \rightarrow o$ -typed non logical constant indexed by the Gödel-number  $\#N_i$  and  $\Delta$  is the  $o$ -typed syntactic parameter indexed by  $\#E_o$ . Thus, the tree  $\text{KrV}(B_{i \rightarrow o})$  is the following (see proof of Main Lemma 7.2, STEP1, point b)):

$$\frac{\frac{\frac{F_{i \rightarrow o} N_i \vdash F_{i \rightarrow o} N_i}{F_{i \rightarrow o} N_i \vdash \exists x_i F_{i \rightarrow o} x_i} \exists\text{-R} \quad \frac{\Delta \vdash B_{i \rightarrow o} w_i}{\Delta \vdash B_{i \rightarrow o} w_i \vee d_o} \vee\text{-R}}{F_{i \rightarrow o} N_i, \Delta \vdash \exists x_i F_{i \rightarrow o} x_i \wedge (B_{i \rightarrow o} w_i \vee d_o)} \wedge\text{-R}}{F_{i \rightarrow o} N_i, \Delta \vdash \exists u_o (\exists x_i F_{i \rightarrow o} x_i \wedge (u_o \vee d_o))} \exists\text{-R}$$

where:  $w_i$  is the  $i$ -typed syntactic parameter indexed by  $\#(\text{KrV}(N_i), \text{KrV}(E_o))$ ;  $d_o$  is the  $o$ -typed non logical constant indexed by  $\#\text{KrV}(E_o)$ . Let us apply the meaning extraction procedure to  $\text{KrV}(B_{i \rightarrow o})$ . At step 0 the sequence of critical chains is:  $\mathcal{T}_1 \equiv \langle (N_i, x_i) \rangle$ ;  $\mathcal{T}_2 \equiv \langle (N_i, x_i), (B_{i \rightarrow o} w_i, u_o) \rangle$  which is the main critical chain of the tree; then, we have  $\mathcal{S}_{01} \equiv \langle N_i \rangle$ ,  $\mathcal{S}_{02} \equiv \langle N_i, B_{i \rightarrow o} w_i \rangle$ . The concatenation sequence  $\mathcal{S}_0$  is  $\langle N_i, N_i, B_{i \rightarrow o} w_i \rangle$ . At step 1 we consider the trees  $\text{KrV}(N_i)$  and  $\text{KrV}(B_{i \rightarrow o} w_i)$  which are elementarily computable (see also Example 10.2) and give:  $\mathcal{S}_{11} \equiv \langle N_i \rangle$ ,  $\mathcal{S}_{12} \equiv \langle B_{i \rightarrow o} w_i \rangle$  so that  $\mathcal{S}_1$  is empty. Thus, the final sequence  $\Sigma$  is  $\langle N_i, N_i, B_{i \rightarrow o} w_i \rangle$  and

$$\text{Meaning}(B_{i \rightarrow o}) \equiv \langle \exists x_i (N_i =_i x_i), \exists x_i (N_i =_i x_i), \forall w_i \exists u_o (u_o \leftrightarrow B_{i \rightarrow o} w_i) \rangle.$$

The tree  $\text{KrV}(C_i)$  is

$$\frac{M_{i \rightarrow o} C_i \vdash M_{i \rightarrow o} C_i}{M_{i \rightarrow o} C_i \vdash \exists x_i M_{i \rightarrow o} x_i} \exists\text{-R}$$

where  $M_{i \rightarrow o}$  is the  $i \rightarrow o$ -typed non logical constant indexed by  $\#C_i$ . The main critical chain is  $\langle (C_i, x_i) \rangle$ . We have that  $\text{KrV}(B_i) * \text{KrV}(C_i)$  is, by definition of  $*$  in the algebra  $\text{Inf-}\mathcal{A}$ :

$$\frac{\Omega \vdash B_{i \rightarrow o} C_i}{\Omega \vdash \exists y_o (y_o)} \exists\text{-R}$$

where  $\Omega$  is the  $o$ -typed syntactic parameter indexed by  $\#B_{i \rightarrow o} C_i$ . Such tree coincides with  $\text{KrV}(B_{i \rightarrow o} C_i)$ , by definition of  $\text{KrV}$  in  $\text{Inf-}\mathcal{A}$ . Therefore,  $\text{Meaning}(B_{i \rightarrow o}) \odot \text{Meaning}(C_i)$  coincides with  $\text{Meaning}(B_{i \rightarrow o})$ . Formally:  $\langle \exists x_i (N_i =_i x_i), \exists x_i (N_i =_i x_i), \forall w_i \exists u_o (u_o \leftrightarrow B_{i \rightarrow o} w_i) \rangle \odot \langle \exists x_i (C_i =_i x_i) \rangle \equiv \langle \exists u_o (u_o \leftrightarrow B_{i \rightarrow o} C_i) \rangle$ .

$\diamond$

In the frames in which the meaning is compositional we can define the notion of *meaning spectrum*, that expresses the semantical complexity of a formula through the meanings of its subformulas. This allows a more refined semantical analysis of  $\omega$ -typed formulas: the spectra comparison between sentences may be more interesting than the simple meaning comparison.

**Definition 10.9.** Let  $\langle (\{\mathbf{K}D_\alpha\}, \mathbf{V}), \Delta, \cong \rangle$  be an arbitrary inferential semantics frame for  $\mathbf{LK}_\omega$  where the meaning is compositional. Let  $B_\alpha$  be any arbitrary closed formula and let  $\mathbf{W}$  be the construction tree of  $B_\alpha$ , having as nodes the subformulas of  $B_\alpha$  (see [18] Ch.9, p.141). Consider the nodes of  $\mathbf{W}$  as ordered in the following way: the tree branches are ordered from the leftmost to the rightmost, and in a same branch the upper node precedes the lower node; assign 1 to the uppermost node in the leftmost branch and proceed without counting any node twice. Write the resulting sequence  $\mathbf{U}$  of the  $B_\alpha$ -subformulas. Then the *Meaning – spectrum* of  $B_\alpha$  is the concatenation of the sequences in  $\mathbf{U}$ .  $\diamond$

Through the *Meaning – spectrum*, in the frames with a compositional meaning, the semantical analysis of any constant  $C_\omega$  of the language may be very different from that of a compound sentence.

**Definition 10.10.** Let  $\langle (\{\mathbf{K}D_\alpha\}, \mathbf{V}), \Delta, \cong \rangle$  be an arbitrary inferential semantics frame for  $\mathbf{LK}_\omega$  and  $B_\alpha$  a closed formula. Let  $\eta$  any syntactic complexity measure on  $\mathbf{LK}_\omega$ -formulas; then a *semantical complexity measure* of  $B_\alpha$  in the given frame is  $\eta(\text{Meaning}(B_\alpha))$   $\diamond$

A semantical complexity measure is therefore recursively computable from the objects  $\mathbf{V}(B_\alpha)$  and  $\text{Meaning}(B_\alpha)$ . It is in general independent of the syntactic complexity of  $B_\alpha$ .

## 11. Conclusions and work in progress

Inferential semantics for classical type theory is the basis for a research program which is in progress, whose main aims are presented here. The program includes the developments of the new formal definition of meaning, the introduction of a logical interpretation of typed  $\lambda$ -calculus, and a general and systematic extension of inferential semantics to constructive logics.

### 11.1. Meaning and semantical complexity of proofs

A notion of meaning of a proof can be investigated. As a possible starting point, we may consider the following definition :

**Definition 11.1.** Let  $\langle (\{\mathbf{K}D_\alpha\}, \mathbf{V}), \Delta, \cong \rangle$  be an arbitrary inferential semantics frame for  $\mathbf{LK}_\omega$  and let  $Q$  be a proof in  $\mathbf{LK}_\omega$ . Then:

- i) If  $\mathcal{R}$  is a Comp-rule in  $\mathbf{P}$  then  $\text{Meaning}(\mathcal{R})$  is the concatenation between the sequences  $\text{Meaning}(t)$ , where  $t$  is the  $\mathcal{R}$ - maximal auxiliary term, and  $\text{Meaning}(B)$  where  $B$  is the  $\mathcal{R}$ -auxiliary proposition. We say that  $\text{Meaning}(\mathcal{R})$  is the *meaning of  $\mathcal{R}$  in the given frame*.
- ii) If  $\mathcal{G}$  is a maximal Comp-ADS in  $\mathbf{P}$ , then  $\text{Meaning}(\mathcal{G})$  is the concatenation, following the order of the ADS, of the meanings of the Comp-rules whose maximal auxiliary terms occur in  $\mathcal{G}$ . We say that  $\text{Meaning}(\mathcal{G})$  is the *meaning of  $\mathcal{G}$  in the given frame*<sup>12</sup>.
- iii)  $\text{Meaning}(\mathbf{P})$  is the concatenation, following the ADS order defined in Definition 10.1 point I), of the meanings of the maximal Comp-ADS's of  $\mathbf{P}$ . We say that  $\text{Meaning}(\mathbf{P})$  is the *meaning of the proof  $\mathbf{P}$  in the given frame*.  $\diamond$

Thus, it will be possible to have proofs with isomorphic or identical meanings in a given frame, and to define semantical complexity measures on proofs, by extending Definition 10.10.

<sup>12</sup>Observe that also the meanings of the auxiliary propositions of the rules, that reflect the context  $\mathbf{P}$ , are included.

### 11.2. A strong inferential semantics for typed $\lambda$ -calculus

The inferential semantics of  $\mathbf{LK}_\omega$  presented in this work also is a semantics for typed  $\lambda$ -calculus. However, it is possible to explore a stronger definition of the inferential interpretation that gives rise to a new kind of model of the typed lambda-calculus, based on proofs. The key point is to introduce inferential algebras having a very expressive bijection  $g$  and to strengthen the notion of semantical identification in an inferential structure. In particular, we are investigating inferential structures  $\langle (\{\text{KrD}_\alpha\}, \text{KrV}), g, \Lambda, \cong \rangle$  such that, having already in each inferential frame  $(V(B_\alpha) \Rightarrow V(C_\beta)) * V(B_\alpha) \equiv V(g(B_\alpha, C_\beta)B_\alpha)$ , it also holds  $\Lambda \vdash g(B_\alpha, C_\beta)B_\alpha =_\beta C_\beta$ . The bijection  $g$  should be defined in a way such that  $g(B_\alpha, C_\beta)$  is  $\lambda x_\alpha C_\beta$ , with  $x_\alpha$  bound variable having as index the Gödel-number of the pair  $\langle B_\alpha, C_\beta \rangle$ . This way, the operation  $\Rightarrow$  of the inferential algebra simulates the  $\lambda$ -abstraction, so that it results as expressed through abstract deduction structures based on the comprehension rules.

### 11.3. Inferential Semantics for intuitionistic and constructive logics

We would like to produce an extension of the inferential semantics approach to the higher order intuitionistic type theory  $\mathbf{LJ}_\omega$  and its subsystems (e.g. the fragments given by uniform proofs and  $\lambda$ -prolog), in a way that fully expresses the constructive character of inferential models. The main goal of the program is the following: to give a uniform method which extracts the inferential semantics of a class  $\{\mathbf{U}_s\}$  of constructive subsystems of  $\mathbf{LK}_\omega$  (among which  $\mathbf{LJ}_\omega$ ) by parametrizing a same schema of model on the system  $\{\mathbf{U}_k\}$  varying in the collection  $\{\mathbf{U}_s\}$ . The idea is to employ the higher order counterpart of the  $\mathbf{LJ}$ -reduction trees  $\{\mathbf{T}_r\}$  à la Takeuti, introduced for the construction of a Kripke-style model for the first order intuitionistic logic  $\mathbf{LJ}$  in [27], pp 57-58. We suppose to associate to each higher order system  $\mathbf{U}$  the corresponding collection  $\{\mathbf{T}_r^U\}$  of higher order Takeuti-like reduction trees, and to construct a Kripke-style inferential model (also recalling [24]) having at nodes suitable inferential frames  $\mathcal{F} \equiv \langle (\{\text{KrE}_\alpha\}_q, \text{KrV}), \Lambda, \cong \rangle$  depending on the reduction tree  $\mathbf{T}^{U,q}$  linked to the node  $q$ .

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## Appendix

### A1 - Example: inferential interpretation in $Inf\text{-}\mathcal{A}$ of the universal quantification

Let  $B_o$  be any  $o$ -typed closed formula in which the subformula  $A_o$  occurs. Here it is shown how the quantification  $\forall x_o B_o$  is represented by the inferential interpretation  $\text{KrV}$  into the inferential algebra  $Inf\text{-}\mathcal{A}$ . The explicit writing is  $\forall_{(o \rightarrow o) \rightarrow o} \lambda x_o B_o$ , that is an application between the terms  $\forall_{(o \rightarrow o) \rightarrow o}$  and  $\lambda x_o [x_o / A_o] B_o$ .

The interpretation  $\text{KrV}(\forall_{(o \rightarrow o) \rightarrow o})$  is the potential proof-tree  $\text{P}(\forall_{(o \rightarrow o) \rightarrow o})$  associated to  $\forall_{(o \rightarrow o) \rightarrow o}$  in the sub-domain  $\text{KrW}_{(o \rightarrow o) \rightarrow o}$  of  $Inf\text{-}\mathcal{P}$ , included in the domain  $\text{KrD}_{(o \rightarrow o) \rightarrow o}$  of  $Inf\text{-}\mathcal{A}$ ;  $\text{P}(\forall_{(o \rightarrow o) \rightarrow o})$  is obtained through iterated abstraction operations  $\Rightarrow$  starting from the basis sub-domain  $\text{KrW}_o$  contained in  $\text{KrD}_o$ . In order to construct such abstractions it is necessary to use the canonical recursive bijection  $g$  (Definition 3.9) associated to the algebra, that is:

$$g_{(o \rightarrow o, o)}^{-1}(\bigvee_{(o \rightarrow o) \rightarrow o}) \equiv (F_{o \rightarrow o}, G_o)$$

$$g_{(o, o)}^{-1}(F_{o \rightarrow o}) \equiv (C_o, E_o)$$

$F_{o \rightarrow o}, G_o, C_o, E_o$  unambiguously and recursively determined closed formulas, given  $\bigvee_{(o \rightarrow o) \rightarrow o}$ .  
Therefore, by the definition of the inferential algebra  $Inf\text{-}\mathcal{A}$  and of the sub-algebra  $Inf\text{-}\mathcal{P}$  it holds:  
 $P(\bigvee_{(o \rightarrow o) \rightarrow o}) \equiv P(F_{o \rightarrow o}) \Rightarrow P(G_o) \equiv (P(C_o) \Rightarrow P(E_o)) \Rightarrow P(G_o)$   
where  $P(C_o), P(E_o), P(G_o)$  are the following potential proof-trees in the inferential domain  $\text{Kr}\mathbf{W}_o$  contained in  $\text{Kr}\mathbf{D}_o$ :

$$\frac{\Gamma \vdash C_o}{\Gamma \vdash \exists x_o(x_o)} \exists\text{-R} \quad \frac{\Delta \vdash E_o}{\Delta \vdash \exists z_o(z_o)} \exists\text{-R} \quad \frac{\Omega \vdash G_o}{\Omega \vdash \exists w_o(w_o)} \exists\text{-R}$$

$x_o, z_o, w_o$  bound variables;  $\Gamma, \Delta, \Omega$  syntactic parameters, respectively indexed by the Gödel-numbers  $\#C_o, \#E_o, \#G_o$ .  
The tree  $P(F_{o \rightarrow o}) \equiv P(C_o) \Rightarrow P(E_o)$  in  $\text{Kr}\mathbf{W}_{o \rightarrow o}$  is the following (see proof of Lemma 7.2, STEP1, point b)):

$$\frac{\frac{\frac{\Gamma \vdash C_o}{\Gamma \vdash C_o \vee h_o} \vee\text{-R}}{\Gamma \vdash \exists x_o(x_o \vee h_o)} \exists\text{-R} \quad \frac{\Delta \vdash F_{o \rightarrow o} y_o}{\Delta \vdash F_{o \rightarrow o} y_o \vee b_o} \vee\text{-R}}{\frac{\Gamma, \Delta \vdash \exists x_o(x_o \vee h_o) \wedge (F_{o \rightarrow o} y_o \vee b_o)}{\Gamma, \Delta \vdash \exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o))} \wedge\text{-R}}{\Gamma, \Delta \vdash \exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o))} \exists\text{-R}$$

where:

$h_o$  non logical constant having as index the Gödel-number  $\#P(C_o)$ ;

$b_o$  non logical constant having as index the Gödel-number  $\#P(E_o)$ ;

$y_o$  is the memory parameter, i.e. the syntactic parameter having as index the Gödel-number  $\#(P(C_o), P(E_o))$ ;

$x_o, u_o$  bound variables;

the *main critical chain* is  $\mathcal{T} \equiv \langle (C_o, x_o), (F_{o \rightarrow o} y_o, u_o) \rangle$  having inferential type  $o \rightarrow o$ ; it also is the main weak critical chain of the tree.

The tree  $P(\bigvee_{(o \rightarrow o) \rightarrow o}) \equiv P(F_{o \rightarrow o}) \Rightarrow P(G_o)$  is the following (see proof of Lemma 7.2, STEP1, point c)):

$$\frac{\frac{\frac{\Gamma \vdash C_o}{\Gamma \vdash C_o \vee h_o} \vee\text{-R}}{\Gamma \vdash \exists x_o(x_o \vee h_o)} \exists\text{-R} \quad \frac{\Delta \vdash F_{o \rightarrow o} y_o}{\Delta \vdash F_{o \rightarrow o} y_o \vee b_o} \vee\text{-R}}{\frac{\Gamma, \Delta \vdash \exists x_o(x_o \vee h_o) \wedge (F_{o \rightarrow o} y_o \vee b_o)}{\Gamma, \Delta \vdash \exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o))} \wedge\text{-R}}{\Gamma, \Delta \vdash \exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o))} \exists\text{-R}}{\frac{\Gamma, \Delta \vdash \exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (F_{o \rightarrow o} y_o \vee b_o))}{\Gamma, \Delta \vdash \exists q_{o \rightarrow o}(\exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (q_{o \rightarrow o} \vee b_o)))} \vee\text{-R}}{\Gamma, \Delta \vdash \exists q_{o \rightarrow o}(\exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (q_{o \rightarrow o} \vee b_o)))} \exists\text{-R}$$

(consider the root of the proof segment above as the left premise of the  $\wedge\text{-R}$  rule below)

$$\frac{\frac{\Omega \vdash \bigvee_{(o \rightarrow o) \rightarrow o} \theta_{o \rightarrow o}}{\Omega \vdash \bigvee_{(o \rightarrow o) \rightarrow o} \theta_{o \rightarrow o} \vee d_o} \vee\text{-R}}{\frac{\Gamma, \Delta, \Omega \vdash \exists q_{o \rightarrow o}(\exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (q_{o \rightarrow o} y_o \vee b_o))) \wedge (\bigvee_{(o \rightarrow o) \rightarrow o} \theta_{o \rightarrow o} \vee d_o)}{\Gamma, \Delta, \Omega \vdash \exists r_o(\exists q_{o \rightarrow o}(\exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (q_{o \rightarrow o} y_o \vee b_o))) \wedge (r_o \vee d_o))} \wedge\text{-R}}{\Gamma, \Delta, \Omega \vdash \exists r_o(\exists q_{o \rightarrow o}(\exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (q_{o \rightarrow o} y_o \vee b_o))) \wedge (r_o \vee d_o))} \exists\text{-R}$$

Note that the left abstracted tree  $P(F_{o \rightarrow o})$  occurs as a leftmost segment of the abstraction tree.

As to the new constants and syntactic parameters note that:

$d_o$  is a non logical constant having as index  $\#P(G_o)$ ;

$\theta_{o \rightarrow o}$  is the memory parameter, i.e. the syntactic parameter having as index  $\#(P(F_{o \rightarrow o}), P(G_o))$ ;

$r_o, q_{o \rightarrow o}$  are bound variables.

The *main critical chain* is :  $\mathcal{T} \equiv \langle (F_{o \rightarrow o}, q_{o \rightarrow o}), (\bigvee_{(o \rightarrow o) \rightarrow o} \theta_{o \rightarrow o}, r_o) \rangle$  with *inferential type*  $(\mathcal{T}) = (o \rightarrow o) \rightarrow o$ .

The interpreted formula  $\bigvee_{(o \rightarrow o) \rightarrow o}$  is the leftmost component of the  $\mathcal{T}$ -theorem.

Now the interpretation  $\text{KrV}(\lambda x_o B_o)$  is computed. It is the potential proof-tree  $P(\lambda x_o B_o)$  associated to  $\lambda x_o B_o$  in the sub-domain  $\text{Kr}\mathbf{W}_{o \rightarrow o}$  of  $Inf\text{-}\mathcal{P}$ , included in the domain  $\text{Kr}\mathbf{D}_{o \rightarrow o}$  of  $Inf\text{-}\mathcal{A}$ :

$P(\lambda x_o B_o)$  is obtained through abstraction operations starting from the basis sub-domain  $\text{Kr}\mathbf{W}_o$ . In order to construct such abstractions, it is necessary to use the canonical recursive bijection  $g$  (Def. 3.9) associated to the algebra, that is:

$$g_{(o, o)}^{-1}(\lambda x_o B_o) \equiv (R_o, S_o)$$

$R_o, S_o$  univocally and recursively determinated closed formulas, given  $\lambda x_o B_o$ .

By the definition of  $Inf-\mathcal{A}$  and  $Inf-\mathcal{P}$  it holds:

$$P(\lambda x_o B_o) \equiv P(R_o) \Rightarrow P(S_o)$$

where  $P(R_o), P(S_o)$  are the following potential proof-trees in the inferential domain  $KrW_o$  contained in  $KrD_o$ :

$$\frac{\Gamma \vdash R_o}{\Gamma \vdash \exists w_o(w_o)} \exists\text{-R} \quad \frac{\Delta \vdash S_o}{\Delta \vdash \exists z_o(z_o)} \exists\text{-R}$$

$w_o, z_o$  bound variables;  $\Gamma, \Delta$  suitably indexed syntactic parameters of the potential proof-trees.

The tree  $P(\lambda x_o B_o) \equiv P(R_o) \Rightarrow P(S_o)$  in  $KrW_{o \rightarrow o}$  is the following (see proof of Lemma 7.2, STEP1, point b):

$$\frac{\frac{\frac{\Gamma \vdash R_o}{\Gamma \vdash R_o \vee p_o} \vee\text{-R}}{\Gamma \vdash \exists w_o(w_o \vee p_o)} \exists\text{-R} \quad \frac{\frac{\Delta \vdash \lambda x_o B_o \mu_o}{\Delta \vdash \lambda x_o B_o \mu_o \vee f_o} \vee\text{-R}}{\Gamma, \Delta \vdash \exists w_o(w_o \vee p_o) \wedge (\lambda x_o B_o \mu_o \vee f_o)} \wedge\text{-R}}{\Gamma, \Delta \vdash \exists u_o(\exists w_o(w_o \vee p_o) \wedge (u_o \vee f_o))} \exists\text{-R}$$

where:

$p_o$  non logical constant having index  $\#P(R_o)$ ;  $f_o$  non logical constant having as index  $\#P(S_o)$ ;  $\mu_o$  is the memory parameter .i.e. the syntactic parameter having as index  $\#(P(R_o), P(S_o))$ ;  $w_o, u_o$  bound variables;

the *main critical chain* is  $\mathcal{T} \equiv \langle (R_o, w_o), (\lambda x_o B_o \mu_o, u_o) \rangle$  having *inferential type*  $o \rightarrow o$ .

The application  $P(\bigvee_{(o \rightarrow o) \rightarrow o}) * P(\lambda x_o B_o)$  recursively produces the tree  $P(\bigvee_{(o \rightarrow o) \rightarrow o} \lambda x_o B_o)$  in  $KrW_o$  of  $Inf-\mathcal{P}$ , which is:

$$\frac{\Lambda \vdash \bigvee_{(o \rightarrow o) \rightarrow o} \lambda x_o B_o}{\Lambda \vdash \exists s_o(s_o)} \exists\text{-R}$$

$\Lambda$  suitably indexed syntactic parameter. Therefore, in the inferential algebra the following holds:

$$KrV(\bigvee_{(o \rightarrow o) \rightarrow o}) * KrV(\lambda x_o B_o) = KrV(\bigvee_{(o \rightarrow o) \rightarrow o} \lambda x_o B_o).$$

## A2

**Lemma 7.8** *Let  $\mathbf{P}^\delta$  be the main module of any arbitrary  $\mathbf{R}^\delta \in KrD_\delta$ ; then either  $\mathbf{P}^\delta$  is in  $\mathbf{R}^\delta \in KrD_\delta$  too, or a suitable replacement of memory parameters, memory constants and syntactic metavariables is always possible in  $\mathbf{P}^\delta$ , so that the resulting tree  $\mathbf{H}$  is in  $\mathbf{R}^\delta \in KrD_\delta$ .*

*Proof:* By induction on types:

a) Primitive types: the main module (see proof of Main Lemma, 7.2 STEP1a)) of each element of  $KrD_o$  (resp.  $KrD_i$ ) is in  $KrD_o$  (resp.  $KrD_i$ ) by the definition of the domain;

b) Basis case for the compound types: let  $\delta \in \{o \rightarrow o, i \rightarrow o, o \rightarrow i, i \rightarrow i\}$  and  $\mathbf{R}^\delta \equiv \mathbf{M}^\alpha \Rightarrow \mathbf{M}^\beta$ ,  $\alpha, \beta \in \{o, i\}$ ; if  $\mathbf{N}^\alpha, \mathbf{N}^\beta$  are the main modules of  $\mathbf{M}^\alpha, \mathbf{M}^\beta$ , they by point a) above are in  $KrD_\alpha, KrD_\beta$  and so  $\mathbf{N}^\alpha \Rightarrow \mathbf{N}^\beta$  belongs to  $KrD_{\alpha \rightarrow \beta}$ ; moreover, by construction, it differs from the main module  $\mathbf{P}$  of  $\mathbf{M}^\alpha \Rightarrow \mathbf{M}^\beta$  only in the indexes of memory constants and memory parameters. Then it can be obtained from  $\mathbf{P}$  by suitable replacements of memory constants and memory parameters, that are always possible.

c) Induction step: let  $\mathbf{R}^\delta \equiv \mathbf{M}^\alpha \Rightarrow \mathbf{M}^\beta$ ; by induction hypothesis, from the main modules  $\mathbf{N}^\alpha, \mathbf{N}^\beta$  of  $\mathbf{M}^\alpha, \mathbf{M}^\beta$  two trees  $\mathbf{H1}$  and  $\mathbf{H2}$  belonging resp. to  $KrD_\alpha$  and  $KrD_\beta$  can be obtained, by suitable replacements of memory constants and memory parameters; then, their abstraction  $\mathbf{R}$  belongs to  $KrD_{\alpha \rightarrow \beta}$ . By construction, it is possible to obtain the tree  $\mathbf{R}$  from the main module of  $\mathbf{M}^\alpha \Rightarrow \mathbf{M}^\beta$  by suitable replacements of memory constants and memory parameters, and this is the thesis.  $\square$

## A3

**Lemma 7.9** *Let  $\mathbf{R}^\delta$  be any element of  $KrD_\alpha$  that coincides with its main module, having an occurrence of the closed formula  $G_\delta$  as the canonical component; then, for each closed  $\delta$ -typed formula  $F_\delta$  different from  $G_\delta$  a tree  $\mathbf{H}^\delta$  in  $KrD_\alpha$  exists, having  $F_\delta$  as the canonical component, so that  $\mathbf{H}^\delta$  is obtained from  $\mathbf{R}^\delta$  by uniformly replacing each term ancestor of the canonical component  $G_\delta$  by  $F_\delta$ , and by suitable replacement of non logical constants and memory parameters.*

*Proof:* by induction on types:

- a) Primitive types: the thesis is straightforward, by definition of primitive domains, but needs the following attention: if  $\text{KrD}_\delta$  is  $\text{KrD}_i$ , when the main chain theorem  $G_i$  is replaced with  $F_i$ , the index of the non logical constants  $F_{i \rightarrow o}$  of the introducing axioms  $F_{i \rightarrow o} G_i \vdash F_{i \rightarrow o} G_i$  (see Def. 7.1.ii) must be changed, in order to obtain a resulting tree that really belongs to  $\text{KrD}_i$ ; moreover, if  $\text{KrD}_\delta$  is  $\text{KrD}_o$ , if  $G_o$  is replaced with  $F_o$  in the introducing axioms  $\Gamma \vdash G_o$ , the index of the syntactic parameter  $\Gamma$  must be changed.
- b) Basis case for the compound types: let  $\delta \equiv \alpha \rightarrow \beta \in \{o \rightarrow o, i \rightarrow o, o \rightarrow i, i \rightarrow i\}$ , let  $\mathbf{R}^\delta \equiv \mathbf{M}^\alpha \Rightarrow \mathbf{M}^\beta$  and let  $G_\delta \equiv g_{(\alpha, \beta)}(B_\alpha, E_\beta), g_{(\alpha, \beta)}$  canonical recursive bijection (Def. 3.9); then  $B_\alpha, E_\beta$  must be the canonical components, which in the primitive cases coincide with the main chain theorems, of  $\mathbf{M}^\alpha, \mathbf{M}^\beta$ . Moreover, if  $\mathbf{R}^\delta$  coincides with its main module, so it must be for  $\mathbf{M}^\alpha, \mathbf{M}^\beta$ . If  $g_{(\alpha, \beta)}^{-1}(F_\delta)$  is the pair  $(Z_\alpha, C_\beta)$ , it is possible to obtain from  $\mathbf{M}^\alpha, \mathbf{M}^\beta$ , by induction hypothesis, through suitable replacements of non logical constants and memory parameters, the trees  $\mathbf{N}^\alpha, \mathbf{N}^\beta$  in  $\text{KrD}_\alpha, \text{KrD}_\beta$ , that have as canonical components resp.  $Z_\alpha$  and  $C_\beta$ . Then  $\mathbf{N}^\alpha \Rightarrow \mathbf{N}^\beta$  belongs to  $\text{KrD}_{\alpha \rightarrow \beta}$  and, by construction, it can be obtained by a suitable replacement of non logical constants and memory parameters, and by replacing  $G_\delta$  with  $F_\delta$ , from  $\mathbf{R}^\delta \equiv \mathbf{M}^\alpha \Rightarrow \mathbf{M}^\beta$ .
- c) Induction step: let  $\mathbf{R}^\delta \equiv \mathbf{M}^\alpha \Rightarrow \mathbf{M}^\beta$ ,  $\alpha, \beta$  arbitrary types; the proof is similar to point b). □