

Proof Theory and Mathematical Meaning of Paraconsistent C-Systems

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Abstract

A proof theoretic analysis and new *arithmetical semantics* are proposed for some paraconsistent **C**-systems, which are a relevant subclass of *Logics of Formal Inconsistency* (**LFI**s) introduced by W. A. Carnielli et al. [8,9]. The sequent versions **BC**, **CI**, **CIL** of the systems **bC**, **Ci**, **Cil** presented in [8,9] are introduced and examined. **BC**, **CI**, **CIL** admits the cut elimination property and, in general, a weakened subformula property. Moreover, a formal notion of *constructive* paraconsistent system is given, and the constructivity of **CI** is proven. Further possible developments of proof-theory and provability logic of **CI**-based arithmetical systems are sketched, and a possible weakened Hilbert's program is discussed. As to the semantical aspects, *arithmetical semantics* interprets **C**-system formulas into Provability Logic sentences of classical Arithmetic **PA** [2,19,15,22]: thus, it links the notion of *truth* to the notion of *provability* inside a *classical* environment. It makes true infinitely many contradictions $B \wedge \neg B$ and falsifies many arbitrarily complex instances of non-contradiction principle $\neg(A \wedge \neg A)$. Moreover, *arithmetical models* falsify both classical logic **LK** and intuitionistic logic **LJ**, so that a kind of metalogical completeness property of **LFI**-paraconsistent logic w.r.t. *arithmetical semantics* is proven. As a work in progress, the possibility to interpret **CI**-based paraconsistent Arithmetic **PACI** into Provability Logic of classical Arithmetic **PA** is discussed, showing the role that **PACI** *arithmetical models* could have in establishing new meta-mathematical properties, e.g. in breaking classical equivalences between consistency statements and reflection principles.

Key words: Proof theory of paraconsistent logic, Foundations of a constructive paraconsistent mathematics, Classical Provability Logic of **PA** as a model of paraconsistency.

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1 Introduction

In this paper we propose a global proof-theoretic analysis of some paraconsistent **C-systems**, which are one of the primary subclasses of the paraconsistent *Logics of Formal Inconsistency* (**LFIs**) and, furthermore, we present a semantics expressing their constructive nature. Such semantics is defined through new *arithmetical models*, where Provability Logic of *classical* Arithmetic is used to interpret paraconsistent logic: thus, e.g., even if it could seem paradoxical, it will make true some contradictions. We deem that both syntax and semantics proposed in these articles make the **C-systems** suitable to support an interesting kind of paraconsistent mathematics.

We recall that *Logics of Formal Inconsistency* (**LFIs**) and their subclass of **C-systems** have been introduced by W.A. Carnielli and other authors ([8–11]), and that their language is the extension of the classical one through a monadic propositional connective $\circ(\cdot)$. The intended meaning of $\circ B$ is “ B is consistent” that is “ $\langle\langle B$ and not $B \rangle\rangle$ does not hold”. Thus, $\circ B$ is a kind of formal translation of a metatheoretic statement at the object language level, as for the provability predicate $\text{Pr}_{\mathbf{T}}(\cdot)$ happens inside arithmetical systems. We call the formulas of the form $\circ B$ *local consistency assertions*. As we shall see, the monadic connective $\circ(\cdot)$ plays an essential role in the new sequent rules introducing paraconsistent negation.

In the first part, the sequent versions **BC**, **CI**, **CIL** of the systems **bC**, **Ci**, **Cil** presented in [8,9] are introduced and examined (Sections 2,3,4,5). As to the reasons of the selection of **bC**, **Ci** and **Cil**¹ in the **C-system** class, we refer to Section 1.1. We show that **BC**, **CI**, **CIL** admit the cut elimination property and, in general, a weakened subformula property. On such results our research program can be founded, aiming to further investigate **C-system** based paraconsistent Arithmetic.

We remark that **bC** is the basic system in the hierarchy of **C-systems** considered here, and the corresponding sequent version **BC** (Section 3) has proper non-classical negation rules on the left side, some of them requiring constraint formulas of the form $\circ F$ in the premise. **bC** is the minimal **C_{min}** plus $\circ A \rightarrow (A \rightarrow (\neg A \rightarrow B))$ [9], and no theorem of **bC** has the form $\circ A$: coherently, no rule of **BC** introduces the connective $\circ(\cdot)$. It is worth noting that both $\circ(\cdot)$ and the propositional negation \neg result as intensional logical operators: this

Gentilini).

¹ **Cil** also belongs to the sub-class of **dC-systems**, since in it the connective $\circ(\cdot)$ can be defined by the standard connectives, i.e. $\neg(B \wedge \neg B) \leftrightarrow \circ B$ is a **Cil**-theorem: this fact makes **Cil** less expressive than **Ci**, since the local consistency assertions $\circ B$ lose their metatheoretic character and the intensionality of $\circ(\cdot)$ collapses into the intensionality of paraconsistent negation.

property has already been proved in [4] through Provability Logic tools, but is explicitly shown at the semantical level by *arithmetical models* in Sections 7, 8. In particular we shall see that the paraconsistent negation has in general a Π_1 -complexity in the classical sense.

The most expressive system in the hierarchy is **Ci**, which is **bC** plus $\neg^\circ A \rightarrow A \wedge \neg A$ and has a relevant set of theorems of the form $^\circ A$. Thus, the corresponding sequent version **CI** (Section 4) has a proper rule introducing $^\circ(\cdot)$. $^\circ B$ is neither **bC**- nor **Ci**-equivalent to $\neg(B \wedge \neg B)$. Then, **CI** maintains the intensional distinction between $^\circ(\cdot)$ and the remaining connectives. The system **Cil** is **Ci** plus $\neg(A \wedge \neg A) \rightarrow ^\circ A$, it is close to the seminal system **C**₁ of Da Costa [12] and defines $^\circ(\cdot)$ as $\neg(B \wedge \neg B) \leftrightarrow ^\circ B$: this fact makes **Cil** less expressive than **Ci**. However, **Cil** has a more powerful negation and the sequent version **CIL** (Section 5) is obtained from **CI** by adding a further constrained negation rule on the left side.

The proof theoretic analysis performed in Sections 3, 4, 5 allows a comparison between **BC**, **CI**, **CIL** and the sequent versions of intuitionistic logic, and leads to an exploration of the notion of *constructivity* for paraconsistent logic. Then, in Section 6, starting from the suggestions that arise from the literature on *duality* between paraconsistency and intuitionism (see, e.g. Brunner-Carnielli [5], Urbas [25], Aoyama [1]) a definition of *constructive* paraconsistent system is proposed, based on the properties of sequent negation rules. A hierarchy in constructivity is given for systems in the **LFI** language, that distinguishes *pseudo-constructive*, *declaratively constructive* and *canonically constructive* paraconsistent systems. To the last grade also a semantical requirement contributes, i.e. the fact that the system admits arithmetical models such that *negation is interpreted as unprovability condition*, that is, in general, the interpretation φ gives $\varphi(\neg B) \equiv \neg \text{Pr}_{\mathbf{PA}}(\varphi(B))$. It is proven that **CI** is a canonically constructive paraconsistent system.

The proof theory of **C**-systems is also the main step to define a paraconsistent Arithmetic admitting proofs with suitable regularity properties, in order to develop both Provability Logic and Proof-Theory of a possible constructive paraconsistent Mathematics founded on *Logics of Formal Inconsistency*. Indeed, as sketched in Section 10, the **CI**-based Arithmetic **PACI**, already introduced in [14], may have new interesting properties as to the **PACI**-unprovability of **PACI**-non triviality. Thus, as discussed in Section 10.1, an **LFI** based *weakened* Hilbert program could be formulated, and a comparison is possible with the proposal by Meyer and Mortensen [20] about a *renewed* Hilbert program based on paraconsistent Relevant Arithmetic **R**.

As to the introduced *arithmetical semantics* (Sections 7, 8) of **BC**, **CI** and **CIL**, the *arithmetical models* falsify both classical logic **LK** and intuitionistic logic **LJ** (Section 9), so that a kind of metalogical completeness property of

LFI-paraconsistent logic w.r.t. *arithmetical models* is proven. We refer to the introduced semantics as a *constructive arithmetical semantics* since it interprets **C**-system formulas into Provability Logic sentences of classical Arithmetic **PA** (see Section 7 and [2,19,15,22]): thus, it links the notion of *truth* to the notion of *provability* inside a *classical* environment. However, it makes true infinitely many arbitrarily complex contradictions $B \wedge \neg B$ and falsifies many arbitrarily complex non-contradiction principle instances $\neg(A \wedge \neg A)$, and so on. Obviously, infinitely many contradictions result as false in arithmetical semantics too: therefore, one could say that it shows the constructive nature of some specific kinds of contradictions $B \wedge \neg B$. In addition, as illustrated in Section 11, arithmetical semantics extended to **PACI** could have a substantial role in order to discover specific meta-mathematical properties that paraconsistent arithmetical theories have w.r.t. classical Arithmetic, breaking some consolidated classical equivalences between important formalized metatheoretic principles.

In this paper we will focus on the definition of arithmetical semantics for the propositional part of the systems **BC**, **CI**, and **CIL**: indeed, the peculiarity of the systems is expressed by the new propositional rules for \neg and $\circ(\cdot)$, and the main task of the semantics is the connotation of \neg and $\circ(\cdot)$ as intensional connectives. The predicative case will be properly treated as a subcase of Arithmetic **PACI**, in the forthcoming work on the interpretation into classical **PA**-Provability Logic of **LFI** based paraconsistent Arithmetic.

1.1 *bC,Ci,Cil: the reasons of a selection*

It is worth noting that in [8] also the systems **mbC**, **mCi**, **mCil** are introduced, that essentially differ from **bC**, **Ci**, **Cil** in the fact that left double negation principle $\neg\neg B \rightarrow B$ is not a theorem. Our preference, as to our main purposes, for **bC**, **Ci**, and **Cil**, is due to the following considerations. We think that in order to define some relevant systems of paraconsistent Arithmetic based on **LFI**s, that must be compared with intuitionistic and classical Arithmetic, we must focus on *few* expressive **C**-systems, and, moreover, we have to stress the peculiarity of the intended paraconsistent Arithmetic w.r.t. intuitionistic Arithmetic. Indeed, we wish to explore the specific constructive character of an **LFI** based Mathematics: therefore, we must avoid any overlap between the paraconsistent and the intuitionistic perspective. Conversely, we wish to emphasize the antisymmetric behaviour w.r.t. negation that such two fundamental non-classical logics show. Moreover, a further reason to choose this strategy is that the notion of *constructivity* for paraconsistent logic introduced in Section 6 is produced by an abstraction of such antisymmetry. Thus, following the seminal definition of Da Costa's system **C**₁ [12], we admit the principles of *left double negation* $\neg\neg B \rightarrow B$ and *excluded middle* $A \vee \neg A$ in the

paraconsistent logics we choose: as wellknown, such principles are discharged by intuitionistic logic. On the other hand, we exclude from our selection any **C**-system that proves the *right double negation* principle $B \rightarrow \neg\neg B$: we assume such principle to be a specific feature of (standard) intuitionistic logic. As to the *non-contradiction principle* $\neg(B \wedge \neg B)$, it is a theorem of intuitionistic logic for each B , whereas only a proper subclass of the instances of such principle are proven by **LFI**s.

1.2 Related works

The main references for *Logics of Formal Inconsistency* are Carnielli-Coniglio-Marcos [8] and Carnielli-Marcos [9]; for an exhaustive overview also Carnielli-Marcos [10,11] and the seminal work of Da Costa [12] are useful. We note that [9] is directly focused on the **C**-systems **bC**, **CI**, **CIi**. For the discussion of the peculiarities of first order **LFI**s we refer to Avron-Zamansky [3]. On the other hand, the present article is aimed to give the logical basis for a constructive Paraconsistent Arithmetic and for the systematic development of Paraconsistent Provability Logic: thus, it is remarkably linked to Benassi-Gentilini [4] and Gentilini [14], where some **C**-system based arithmetical systems are firstly proposed, and the intensional character of paraconsistent negation is shown through provability logic tools. Since *arithmetical semantics* developed in Sections 7, 8 uses classical Provability Logic of **PA** as a kind of interpretation domain, we refer to Artemov-Beklemishev [2], Japaridze-de Jongh [19] and Smorynski [22] as canonical overviews on the field. As to the proof-theoretic approach to Provability Logic, that provides some tools for Paraconsistent Provability Logic, we mention Gentilini [15–18]. Our reference for basic proof-theory and cut-elimination is Buss [6,7], with some further devices from Takeuti [23] and Troelstra-Schwichtenberg [24]. As to the duality between intuitionism and paraconsistency, from which we start in Section 6 for the definition of constructivity for paraconsistent logic, we refer to Brunner-Carnielli [5] for a more semantical approach, and to Urbas [25] and Aoyama [1] for a sequent based approach. As to the perspective of a conjectural reasoning inside **LFI** based Arithmetic mentioned in Section 11.2, it must be noted that in Forcheri-Gentilini [13] a constructive conjectural calculus is introduced, emphasizing the deduction features of **BC** and **CI**, but the formalization allowed by *constructive conjectures* inside **PACI** would be much stronger, without using labelled formulas.

2 Sequent formalism for the paraconsistent setting

For the language and the hilbertian formulation of **C**-systems we refer to [9,8]. The main novelty is the introduction of the connective $\circ(\cdot)$, such that the intended meaning of $\circ B$ is “ B is consistent” that is “ $\langle\langle B$ and not $B\rangle\rangle$ does not hold”. We call *local consistency assertion* or *circled formula* any formula of the form $\circ F$. B is a *classical formula* if the connective $\circ(\cdot)$ does not occur in B . B is a *positive classical formula* if neither \neg nor $\circ(\cdot)$ occur in B . By provability logic tools we have shown in [4] that \neg and $\circ(\cdot)$ are intensional connectives, and such property will be explicitly declared by the semantics introduced in Section 7. We shall present here the sequent version of the systems **bC**, **Ci**, **Cil** defined in [9]. Recall that a sequent S (see [6,7,23,24]) is an expression of the form $X \vdash Y$ where X and Y are finite (possibly empty) *multisets* of formulas. *Multisets* are sets with “multiplicity”, such that, e.g. $\{A, B, A\}$ and $\{A, B\}$ are different multisets, but $\{A, B\}$ and $\{B, A\}$ are the same multiset (see [24] pag. 5). X is called the *antecedent* of S , Y the *succedent* of S . We will use the symbols $X, Y, \Lambda, \Gamma, \dots$ as meta-expressions for multisets of formulas, A, B, C, D, \dots for formulas. The writing Λ, Γ stands for $\Lambda \cup \Gamma$ and then $A, X \vdash Y$ is an abbreviation of $\{A\} \cup X \vdash Y$. So that, even if we briefly say that “ A occurs in the sequent $A, X \vdash Y$ ”, we must recall that, in a sequent, formulas never occur but multisets only occur. It is useful to establish a priori a clear correspondence between the sequent formulation and the hilbertian formulation of a system: indeed, in a paraconsistent setting the matter is not so obvious as in the classical case. Thus:

the sequent $A_1, A_2, \dots, A_n \vdash B_1, B_2, \dots, B_m$
corresponds to the formula $\bigwedge_i A_i \longrightarrow \bigvee_j B_j$ and viceversa;
the sequent $\vdash B_1, B_2, \dots, B_m$ corresponds to the formula $\bigvee_j B_j$ and viceversa;
the sequent $A_1, A_2, \dots, A_n \vdash$
corresponds to the set of formulas $\{\bigwedge_i A_i \longrightarrow F : F \text{ formula of the language}\}$
and viceversa;
the sequent “ \vdash ” corresponds to the set of formulas

$$\{F : F \text{ formula of the language}\}$$

and viceversa.

Given a rule $\frac{S_1 \dots S_n}{S}$, the sequents S_1, \dots, S_n are the *premises* of the rule, the sequent S is the *conclusion* of the rule. The proofs are trees, whose leaves are *axioms*, and whose branches are formed by sequent rule occurrences. Any root of a tree in a sequent formulated system is a *theorem* of the system. We moreover say that any formula A is a theorem of a sequent system **T** if the sequent $\vdash A$ is a theorem of **T**. We say that a sequent formulated system or theory *trivializes* (or *is trivial*) if and only if it proves each sequent of the

form $\vdash A$. A sequent formulated theory \mathbf{T} has the *bottom particle property* if it proves any sequent of the form $B \vdash$, where B is called a *bottom particle* for \mathbf{T} . If \mathbf{T} has the bottom particle property we say that \mathbf{T} is trivial if and only if it proves the empty sequent “ \vdash ”. \mathbf{T} is *negation consistent* if it cannot prove any formula of the form $B \wedge \neg B$, *negation inconsistent* otherwise. In writing formulas we adopt the convention that \vee, \wedge, \neg link more than \rightarrow , and that \rightarrow links more than \leftrightarrow .

Some remarks must be pointed out on the quantifier rules that should be added to the propositional rules of the sequent versions of **bC**, **CI**, **CIl** in order to get the corresponding first order predicate systems **BC**, **CI**, **CIL**. The following set of standard quantifier rules² is necessary:

$$\frac{[t/x] A, \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} \forall - L \quad \frac{\Gamma \vdash \Delta, [b/x] A}{\Gamma \vdash \Delta, \forall x A} \forall - R$$

$$\frac{[b/x] A, \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta} \exists - L \quad \frac{\Gamma \vdash \Delta, [t/x] A}{\Gamma \vdash \Delta, \exists x A} \exists - R$$

but it is not enough. Indeed, as discussed in Avron-Zamansky [3], further rules should be added to avoid some abnormalities in the proof capabilities of the systems, that would have no substantial meaning from the standpoint of paraconsistency, and would be simply unacceptable. For example, in these systems $\forall x A(x) \leftrightarrow \forall y A(y)$ is provable, but $\neg \forall y A(y) \leftrightarrow \neg \forall x A(x)$ is not, and $\forall x \forall y A(x) \leftrightarrow \forall x A(x)$ is provable, but $\neg \forall x A(x) \leftrightarrow \neg \forall x \forall y A(x)$ is not. Therefore, some *conversion rules*, identifying formulas that differ only in the names of bound variables or in vacuos quantifier occurrences, should be added. However, having considered that the proof theoretic properties of **BC**, **CI**, **CIL**, and the demonstrations of the cut elimination theorems (Sections 3,4,5), essentially depend on the propositional rules, and that arithmetical semantics (Section 8) is mainly focused on the propositional connectives \neg and $\circ(\cdot)$, we will examine in this article the sequent systems **BC**, **CI**, **CIL** only in their propositional parts. Moreover, for the sake of simplicity, we will maintain the same names for the propositional parts and their first order extensions. The predicate cases will be treated in a next work dedicated to **CI**-based Arithmetic **PACI**: in such context, the study of first order properties of **BC**, **CI**, **CIL** acquires a strong motivation, i.e. the proof theoretic and semantical analysis of **LFI** based paraconsistent Arithmetic.

² Where t is an arbitrary term and b is a free variable which does not occur in Γ, Δ , and t may be not fully quantified while b must be uniformly replaced by x , see [23].

3 The system **BC**

In this section we give an exhaustive exposition of the proof-theory of **BC**, corresponding to the Hilbert formulated **C**-system **bC**. Some results on **BC** have been already presented in [4], but the proofs were only outlined there, since they were not the main focus of that paper. Conversely, a detailed proof of cut-elimination for **BC** is presented here, since it defines and produces some tools which are necessary for the results on **CI** and **CIL**, that are proven in this paper for the first time.

The sequent system **BC** is the minimal system which expresses some relevant properties of the connective $\circ(\cdot)$ and that owns remarkably powerful negation rules. Observing the structure of **BC**-rules an important consideration arises, that will be confirmed by **CI**- and **CIL**-rules: *paraconsistency lies on the peculiarity of negation rules introducing a negated formula on the left side*, in this case the $\neg - L1$ and $\neg - L3$ rules. This remark will be developed in Section 6. The negation rule on the right side is the classical $\neg - R$. As expected, since **bC** has not theorems of the form $\circ F$, **BC** has not rules introducing $\circ(\cdot)$; however, the negation rule $\neg - L3$ with principal formula $\neg A$ has the formula $\circ A$ as a *constraint formula* in the premise antecedent.

BC is given by:

BC–*Axioms*: $A \vdash A$

BC–*Positive propositional logical rules*:

$$\begin{array}{c} \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge -L \quad \frac{B, \Gamma \vdash \Delta}{B \wedge A, \Gamma \vdash \Delta} \wedge -L \quad \frac{\Gamma \vdash \Delta, A \quad \Lambda \vdash X, B}{\Gamma, \Lambda \vdash \Delta, X, A \wedge B} \wedge -R \\ \\ \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \vee -R \quad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, B \vee A} \vee -R \quad \frac{A, \Gamma \vdash \Delta \quad B, \Lambda \vdash X}{A \vee B, \Gamma, \Lambda \vdash \Delta, X} \vee -L \\ \\ \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B} \rightarrow -R \quad \frac{\Gamma \vdash \Delta, A \quad B, \Lambda \vdash X}{A \rightarrow B, \Gamma, \Lambda \vdash \Delta, X} \rightarrow -L \end{array}$$

BC–*Negation rules*:

$$\begin{array}{c} \frac{A, \Gamma \vdash \Delta}{\neg \neg A, \Gamma \vdash \Delta} \neg - L1 \quad \frac{\circ A, \Gamma \vdash \Delta, A}{\circ A, \neg A, \Gamma \vdash \Delta} \neg - L3 \\ \\ \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \neg - R \end{array}$$

The formula $\circ A$ in the rule $\neg - L3$ will be called the *constraint formula* of the rule $\neg - L3$.

BC–*Structural rules*:

Contraction rules:

$$\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} C - R \quad \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} C - L;$$

Weakening rules:

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} W - R \quad \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} W - L;$$

$$\text{Cut rule: } \frac{\Gamma \vdash \Delta, A \quad A, \Lambda \vdash X}{\Gamma, \Lambda \vdash \Delta, X} \text{Cut}$$

We note that the use of *multisets* of formulas instead of simple *sets* of formulas allows to consider $A, A, \Gamma \vdash \Delta$ and $A, \Gamma \vdash \Delta$ as different sequents. However, we immediately infer the latter from the former by a contraction, and the former from the latter by a weakening. For the sake of simplicity we establish the following convention: *in the sequel, in general, the application of contraction rules will be not explicitly mentioned or indicated. We assume that a contraction rule is applied each time it is necessary, given the context of the discourse. The same holds as to the contraction elimination.*

The rule $\neg - L3$ expresses the peculiarity of paraconsistent negation and the link between the paraconsistent negation and the connective $\circ(\cdot)$. As already pointed out, both connectives can be seen as intensional connectives: for example, in [4] it is shown that in the **BC**-based Arithmetic **PCA** the negation $\neg B$ of a Δ_0 -formula B is not a Δ_0 -formula.

The *classical predicate calculus* **LK** [7,23] can be obtained from **BC** by replacing the pair $\neg - L3, \neg - L1$, with the rule: $\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \neg - L2$. The rules $\neg - L3, \neg - L1$ are also called *proper BC-rules*; we also call *classical rules* the remaining **BC**-rules. The *intuitionistic predicate calculus* **LJ** [23] can be obtained from **LK** by imposing that each sequent in a proof tree has at most one formula in the succedent. This is essentially a constraint on the classical negation rule on the right $\neg - R$, to discharge the possibility to prove all the sequents $\vdash A, \neg A$. Thus, sequent formalism shows an antisymmetric behaviour of paraconsistent systems with respect to intuitionistic systems, as to the structure of negation rules: paraconsistent negation rules are obtained by constraining *classical negation on the left* $\neg - L2$; intuitionistic negation rules are obtained by constraining *classical negation on the right* $\neg - R$, which is restricted to the cases where the premise succedent is empty. In Section 6 we will connect this fact with the already investigated *duality* between intuitionism and paraconsistency (see, e.g., Brunner-Carnielli [5]) and with a notion of

constructivity for paraconsistent logic.

We recall that the Hilbert formulated system **bC** is given by the system \mathbf{C}_{min} (see [9]) *plus* the axiom schema $\circ A \rightarrow (A \rightarrow (\neg A \rightarrow B))$. \mathbf{C}_{min} is given by positive classical logic *plus* the excluded middle principle $A \vee \neg A$ *plus* the left double negation principle $\neg\neg A \rightarrow A$, with the rule of *Modus Ponens*. The sequent version **C – MIN** of \mathbf{C}_{min} is given by **LK** *minus* $\neg\neg L2$ *plus* $\neg\neg L1$. The system **C – MIN** *plus* the sequent $\circ A, \neg A, A \vdash$ corresponds to **bC**, as it follows by the relation between sequent formulation and Hilbert formulation of a system established in Section 2.

Proposition 1 *BC is a sequent version of the system bC presented in [9].*

Proof. We have in **BC**:
$$\frac{A \vdash A}{\frac{\circ A, A \vdash A}{\circ A, A, \neg A \vdash}}$$

On the other hand, from $\circ A, \Gamma \vdash \Delta, A$ we have by cut, in **C – MIN** *plus* $\circ A, A, \neg A \vdash$, the sequent $\circ A, \neg A, \Gamma \vdash \Delta$. ■

We need to recall some notions of proof-theory; our definitions are similar but not identical to Buss [7], Troelstra-Schwichtenberg [24] and, subordinately, to Takeuti [23].

Definition 1 *Let \mathbf{U} be a system of the set $\{\mathbf{LK}, \mathbf{BC}, \mathbf{CI}, \mathbf{CIL}\}$. Then: i) The depth or height $h(P)$ of a tree P in \mathbf{U} is the highest number of proof-lines in a branch. The grade $g(A)$ of a formula A is the number of occurrences of logical symbols in it. ii) Let \mathcal{C} be any cut rule occurrence in a tree in \mathbf{U} . Then the cutrank of \mathcal{C} is the grade of the cut formula in \mathcal{C} , the level of \mathcal{C} is the sum of the depths of the deductions of the premises.* ■

Definition 2 *Let \mathbf{U} be a system of the set $\{\mathbf{LK}, \mathbf{BC}, \mathbf{CI}, \mathbf{CIL}\}$. Then: i) In a rule occurrence \mathcal{R} in a proof-tree P in \mathbf{U} we call: auxiliary formulas the formula occurrences in the premises on which the rule acts; principal formula, the formula occurrence produced by the rule in the conclusion. Each formula in the conclusion of \mathcal{R} is called the successor of the formulas in the premises corresponding to it, that are called its predecessors.*

ii) *In a branch of a proof-tree P in \mathbf{U} we say that the formula occurrence B is an ancestor of the formula occurrence C occurring below B in the branch, called a descendant of B , if they are connected by a sequence of predecessor-successor relations alongside the branch. C is called an integral descendant of B if B and C are the same formula; if C is an integral descendant of B then B is called a direct ancestor of C . If B has a direct ancestor which is the principal formula of an inference \mathcal{R} different from a contraction, or which occurs in an axiom S , then we say that B is introduced by \mathcal{R} (resp. by S) in*

P . ■

Observe that, due to the action of contractions, a formula occurrence A may be introduced in P by a number of different rules or axioms.

It is wellknown that the effective cut-elimination procedure for classical logic **LK** can be described by formulas of *Primitive Recursive Arithmetic* **PRA**. Thus we can consider it as a mathematical object.

Lemma 1 *Let P be any proof in **BC** of the sequent $X \vdash Y$. Then, a suitable **PRA**-formulation \mathcal{F} of the cut reductions prescribed by the cut-elimination procedure for **LK** (see [7],[24]) of cuts having both cut-formulas which are the integral descendants of principal formulas of **LK**-rules or axioms exists, such that \mathcal{F} can be applied to P , obtaining a **BC**-proof P' of $X \vdash Y$.*

Proof. We have to prove that the application of \mathcal{F} to P does not break any proper rule constraint. Let \mathcal{C} be an uppermost cut-occurrence in P :

$$\frac{\frac{Q1 \quad \Gamma \vdash \Delta, A}{\Gamma, \Lambda \vdash \Delta, X} \quad \frac{Q2 \quad A, \Lambda \vdash X}{\Gamma, \Lambda \vdash \Delta, X}}{\Gamma, \Lambda \vdash \Delta, X} c$$

.....

where $Q1$ and $Q2$ are the P -sub proofs of the premises. By hypotheses no cut formula has a direct ancestor introduced by a proper **BC**-rule. Consider the following cases:

1. The cut formula A has not the form ${}^\circ E$. Then, consider the following sub-cases:

1.1 The canonical reductions prescribed by \mathcal{F} in order to allow the induction on the cut level do not perturb any $\neg - L3$ constraint. In fact, no direct ancestor of a cut-formula can be the constraint formula of a $\neg - L3$ rule, so that the cut of lower level prescribed by \mathcal{F} can be applied in the standard way. Indeed, the standard \mathcal{F} -step for reducing a cut \mathcal{C} of the form:

$$\frac{\frac{\Gamma \vdash \Delta, B \quad \frac{B, \Lambda \vdash X, \mathcal{R}}{B, U \vdash V}}{\Gamma, U \vdash \Delta, V} c}{\Gamma, U \vdash \Delta, V} c$$

.....

is to produce the following new cut with a lower level:

$$\frac{\frac{\Gamma \vdash \Delta, B \quad B, \Lambda \vdash X}{\Gamma, \Lambda \vdash \Delta, X} \quad \mathcal{R}}{\Gamma, U \vdash \Delta, V}$$

and the fact that \mathcal{R} may be a $\neg - L3$ rule is not relevant when B has not the form ${}^\circ E$.

1.2 The canonical reductions prescribed by \mathcal{F} in order to allow the induction on the cutrank do not perturb any $\neg - L3$ constraint, even if the auxiliary formulas of the inferences introducing the cut premises have the form ${}^\circ E$.

Consider for example the following cut:

$$\frac{\frac{A, \Gamma \vdash \Delta, {}^\circ B}{\Gamma \vdash \Delta, A \rightarrow {}^\circ B} \quad \frac{\Lambda \vdash X, A \quad {}^\circ B, U \vdash W}{A \rightarrow {}^\circ B, \Lambda, U \vdash X, W}}{\Gamma, \Lambda, U \vdash \Delta, X, W}$$

and assume that ${}^\circ B, U \vdash W$ is the conclusion of a $\neg - L3$ rule having the predecessor of the ${}^\circ B$ -occurrence which is the $\rightarrow -L$ right auxiliary formula as constraint formula. The reduction prescribed by \mathcal{F} for the induction on the cutrank introduces two new cuts with a lower rank:

$$\frac{\frac{\Lambda \vdash X, A \quad A, \Gamma \vdash \Delta, {}^\circ B}{\Gamma, \Lambda \vdash \Delta, X, {}^\circ B} \quad {}^\circ B, U \vdash W}{\Gamma, \Lambda, U \vdash \Delta, X, W}$$

and the mentioned $\neg - L3$ occurrence having ${}^\circ B, U \vdash W$ as conclusion is not perturbed.

2. The cut formula A has the form ${}^\circ E$, and some direct ancestors of the right cut formula may be constraint formulas of possible $\neg - L3$ occurrence in $Q2$. Then, consider the following sub-cases:

2.1 Assume that both cut-formulas *are not* introduced by axioms too. Then, both cut-formula occurrences have all direct ancestors introduced in P by weakenings, since no **LK**-logical rule has ${}^\circ E$ as principal formula. Therefore, if we in $Q1$ delete all such weakenings, that must be of the form $W - R$, we get a cut-free proof of $\Gamma \vdash \Delta$, without breaking any $\neg - L3$ constraint, since $\neg - L3$ constraint formulas occur in the antecedent only.

2.2. Assume that cut formulas are introduced by axioms too. Suppose that the left cut formula also has direct ancestors occurring in axioms $A \vdash A$ of $Q1$. Then, we replace each $A \vdash A$ with the $Q2$ -root $A, \Lambda \vdash X$ and, after deleting the weakenings $W - R$ introducing in $Q1$ the remaining direct ancestors, we get a cut-free proof of $\Gamma, \Lambda \vdash \Delta, X$. ■

Theorem 1 *The system **BC** admits cut-elimination.*

Proof. We consider an uppermost cut \mathcal{C} in a proof-tree P in **BC** and show that it can be replaced by **BC**-deductions without cuts. The proof is by induction on the cutrank with a subinduction on the level of \mathcal{C} . Let \mathcal{C} be the following cut:

$$\frac{\frac{Q1}{\Gamma \vdash \Delta, A} \quad \frac{Q2}{A, \Lambda \vdash X}}{\Gamma, \Lambda \vdash \Delta, X} \quad c$$

.....

We previously establish the following preliminary reductions on the tree P : if one of the cut-formulas A of \mathcal{C} is the integral descendant of the principal formula of a weakening rule in P , and no ancestor of A is the constraint formula of a $\neg - L3$ rule in P , we delete such weakening. If one of the cut-formulas A of \mathcal{C} has all the uppermost ancestors in P introduced by weakenings, and no ancestor of A is the constraint formula of a $\neg - L3$ rule in P , we delete such weakenings, getting a cut-free proof of a sub-sequent of $\Gamma, \Lambda \vdash \Delta, X$.

If the cut formulas of \mathcal{C} can be included in the hypotheses of Lemma 1 we apply the cut reduction allowed by Lemma 1. In particular, the cases where the cut formula has the form ${}^\circ F$ have already been considered at point 2 of the proof of Lemma 1.

Then, we have to consider the cases where at least one direct ancestor of a cut-formula is the principal formula of a proper **BC**-rule, so that A must have the form $\neg B$. Consider these sub-cases:

1. At least one of the sub-proofs $Q1, Q2$ is an axiom. Let the left premise be an axiom of the form $A \vdash A$. Then, the conclusion of \mathcal{C} has the form: $A, \Lambda \vdash X$. Then, we replace \mathcal{C} by the right premise. If the right premise is an axiom, we replace the cut by the left premise.

2. Neither $Q1$ nor $Q2$ is an axiom and the cut formula is not principal in at least one of the premises. If the cut formula is not on the both sides principal, let us consider for example the premise $Z \vdash W, A$ of the one-premise rule \mathcal{R} in $Q1$ having the left cut premise $\Gamma \vdash \Delta, A$ as conclusion. Then we produce the following proof:

$$\frac{\frac{Z \vdash W, A \quad A, \Lambda \vdash X}{Z, \Lambda \vdash W, X} \quad \mathcal{R}}{\Gamma, \Lambda \vdash \Delta, X},$$

where the level of the introduced cut is lower than that of \mathcal{C} , and the cutrank is the same. If \mathcal{R} is a two-premise rule the reduction is similar, and so it is for the sub-cases in which the right cut formula is not principal. Note that since A has the form $\neg B$, no direct ancestor of A can be the constraint formula of

any $\neg - L3$ rule.

3. The cut formula $\neg B$ is principal in both the premises of the cut \mathcal{C} . By hypotheses, both occurrences of $\neg B$ are the principal formulas of negation rules.

3.1 The left premise of \mathcal{C} is the conclusion of a $\neg - R$ rule and the right premise is the conclusion of a $\neg - L1$ rule:

$$\frac{\frac{\neg B, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg \neg B} \quad \frac{B, \Lambda \vdash X}{\neg \neg B, \Lambda \vdash X}}{\Gamma, \Lambda \vdash \Delta, X}.$$

Then, we replace the cut \mathcal{C} with the following proof:

$$\frac{\frac{B, \Lambda \vdash X}{\Lambda \vdash X, \neg B} \quad \neg B, \Gamma \vdash \Delta}{\Gamma, \Lambda \vdash \Delta, X},$$

where the cutrank is lower.

3.2 The left premise of \mathcal{C} is the conclusion of a $\neg - R$ rule and the right premise is the conclusion of a $\neg - L3$ rule:

$$\frac{\frac{B, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg B} \quad \frac{\circ B, W \vdash X, B}{\circ B, \neg B, W \vdash X}}{\circ B, \Gamma, W \vdash \Delta, X}.$$

Then we replace the cut \mathcal{C} with the following proof:

$$\frac{\circ B, W \vdash X, B \quad B, \Gamma \vdash \Delta}{\circ B, \Gamma, W \vdash \Delta, X},$$

where the cutrank is lower. ■

We have a standard subformula property for **BC**.

Proposition 2 *Let P be a cut-free **BC**-proof. Then, if a formula A occurs in P , it also occurs in the P -root.*

Proof. The conclusions of the proper **BC**-rules have the subformula property w.r.t the respective premises. Therefore, the property holds as for the classical logic **LK**. ■

A further remark is that in the proofs of **BC**, *logical axioms cannot be reduced to the atomic case*, as in the classical logic **LK** happens. Indeed, by cut-elimination we see that, *in general*, any sequent $\neg B \vdash \neg B$ cannot be proved

in **BC** by employing atomic axioms only, and the same holds for **CI** and **CIL**.

Proposition 3 **BC** has the bottom particle property.

Proof. Consider the **BC**-theorem $\circ A \wedge \neg A \wedge A \vdash \cdot$. Thus, $\circ A \wedge \neg A \wedge A$ is a bottom particle for **BC** for any A . Moreover, for each arbitrary pair A, F of formulas $\circ A \wedge \neg A \wedge A \wedge F$ is a bottom particle for **BC**, and, by $\neg - L1$, $\neg^{2k}(\circ A \wedge \neg A \wedge A \wedge F)$ is a bottom particle too for each k . ■

4 The system **CI**

As we know from [9,8] **Ci** can be axiomatized as **bC** plus $\neg \circ A \rightarrow A \wedge \neg A$. We consider **Ci**, and then the corresponding sequent system **CI** proposed here, as the most interesting **C**-system: it proves relevant properties of the connective $\circ(\cdot)$ without producing any equivalence between $\circ B$ and any classical formula. On the other hand, it has strong negation rules, even if it remains acceptably away from classical logic **LK**. **CI** has the rule $R\text{Ci}$ that properly introduces the connective $\circ(\cdot)$. Differently, no rule of **BC** introduces $\circ(\cdot)$. As shown in [9] in **CI** an elegant form of *strong negation* \sim can be defined as $\sim A \leftrightarrow (\neg A \wedge \circ A)$, that behaves as a classical negation: as a consequence, classical inference of **LK** can be encoded into **CI** by a map t from **LK**-proofs to **CI**-proofs such that $t(\neg B)$ is $\sim t(B)$ (see [9] p. 51).

The system **CI** is given by adding to **BC** the following *proper CI-rules*:

$$\frac{\Gamma \vdash \Delta, \circ A}{\neg \circ A, \Gamma \vdash \Delta} \neg - L4 \qquad \frac{A \wedge \neg A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \circ A} R\text{Ci}$$

In the rule $R\text{Ci}$ the formula $A \wedge \neg A$ in the premise antecedent is the *$R\text{Ci}$ -auxiliary formula*, the formula $\circ A$ in the conclusion succedent is the *$R\text{Ci}$ -principal formula*. Note that the $\neg - L4$ rule is the restriction to circled formulas of the classical rule $\neg - L2$ for the negation on the left side.

Proposition 4 **CI** is a sequent version of the system **Ci** presented in [9].

Proof. Recall that **Ci** is **bC** plus $\neg \circ A \rightarrow A \wedge \neg A$. **CI** proves $\neg \circ A \vdash A \wedge \neg A$:

$$\frac{\frac{\frac{A \vdash A}{A \wedge \neg A \vdash A} \wedge -L}{\vdash A, \circ A} \wedge -L}{\neg \circ A \vdash A} \neg - L4 \quad R\text{Ci} \qquad \frac{\frac{\frac{\neg A \vdash \neg A}{A \wedge \neg A \vdash \neg A} \wedge -L}{\vdash \neg A, \circ A} \wedge -L}{\neg \circ A \vdash \neg A} \neg - L4 \quad R\text{Ci}$$

from which, by $\wedge - R$, we have $\neg \circ A \vdash A \wedge \neg A$ through a cut-free proof.

Conversely, $\mathbf{V} \equiv \mathbf{BC}$ plus $\neg^\circ A \vdash A \wedge \neg A$ has the same theorems as \mathbf{CI} :

i) \mathbf{V} proves each conclusion of any $\neg - L4$ rule. Assuming any sequent of the form $\Gamma \vdash \Delta, \circ A$ we have:

$$\frac{\Gamma \vdash \Delta, \circ A \quad \frac{\neg^\circ A \vdash A \wedge \neg A \quad \circ A, A \wedge \neg A \vdash}{\neg^\circ A, \circ A \vdash}}{\neg^\circ A, \Gamma \vdash \Delta}$$

where in the uppermost cut the right premise is a \mathbf{BC} -theorem.

ii) \mathbf{V} proves each conclusion of any RCi -rule: from any sequent of the form $A \wedge \neg A, \Gamma \vdash \Delta$, the axiom $\neg^\circ A \vdash A \wedge \neg A$ immediately gives by cut the sequent $\neg^\circ A, \Gamma \vdash \Delta$, from which, by $\neg - R$, the sequent $\Gamma \vdash \Delta, \neg \neg^\circ A$ is derived. Then, by the \mathbf{BC} -theorem $\neg \neg^\circ A \vdash \circ A$, we get $\Gamma \vdash \Delta, \circ A$. ■

Lemma 2 *Let Q be a cut-free proof in \mathbf{CI} of the sequent $A \wedge \neg A, X \vdash Y$. Then we can get from Q a cut-free \mathbf{CI} -proof Q' of $A, \neg A, X \vdash Y$.*

Proof. We delete each weakening introducing any direct ancestor of $A \wedge \neg A$ occurring in the root-antecedent. Thus, we can suppose that such $A \wedge \neg A$ is introduced by axioms or by $\wedge - L$ rules only. Remark that, by construction, no direct ancestor of $A \wedge \neg A$ can be the auxiliary formula of a RCi -rule. Then, we delete in Q each $\wedge - L$ rule introducing any $A \wedge \neg A$ -direct ancestors, and replace each mentioned axiom $A \wedge \neg A \vdash A \wedge \neg A$ with the following proof:

$$\frac{A \vdash A \quad \neg A \vdash \neg A}{A, \neg A \vdash A \wedge \neg A}$$

Since no constraint of any \mathbf{CI} -rule is broken, we get a proof Q' of $A, \neg A, X \vdash Y$, by possibly adding suitable weakenings. ■

Lemma 3 *Let P be any proof in \mathbf{CI} of the sequent $X \vdash Y$. Then, a suitable **PRA**-formulation \mathcal{G} of the cut reductions prescribed by the cut-elimination procedure for \mathbf{BC} (Theorem 1) of cuts having both cut-formulas which are the integral descendants of principal formulas of \mathbf{BC} -rules or axioms exists, such that \mathcal{G} can be applied to P obtaining a \mathbf{CI} -proof P' of $X \vdash Y$.*

Proof. We recall the proof of Lemma 1 and see that the considerations produced there for \mathbf{LK} , \mathcal{F} , \mathbf{BC} can be extended to similar considerations for \mathbf{BC} , \mathcal{G} , \mathbf{CI} . The following remarks are useful:

1. The $\neg - L4$ rule is a subcase of the classical $\neg - L2$ rule of \mathbf{LK} .
2. No cut formula in P may have a direct ancestor which is the auxiliary formula of a RCi -rule.

Therefore, the reductions prescribed by \mathcal{G} do not collide with **CI** proper rule occurrences in P . ■

Theorem 2 *The system **CI** admits cut-elimination.*

Proof. We consider an uppermost cut \mathcal{C} in a proof-tree P in **CI** and show that it can be replaced by **CI**-deductions without cuts. The proof is by induction on the cutrank with a subinduction on the level of \mathcal{C} .

Let \mathcal{C} be the following cut:

$$\frac{\begin{array}{c} Q1 \\ \hline \Gamma \vdash \Delta, A \end{array} \quad \begin{array}{c} Q2 \\ \hline A, \Lambda \vdash X \end{array}}{\hline \Gamma, \Lambda \vdash \Delta, X} \dots\dots$$

where $Q1$ and $Q2$ are the P -sub proofs of the premises.

We establish the preliminary reductions on the tree P that are already mentioned in the proof of the cut elimination theorem for **BC** (Theorem 1), and moreover the following convention: each $\neg - L3$ rule in P having a formula ${}^\circ F$ as auxiliary formula is considered as a $\neg - L4$ rule, so that it *has not* constraint formula.

By Lemma 3, the cases that we have to examine are such that at least one occurrence of the cut formula A also is the integral descendant of the principal formula of a proper **CI**-rule. Consider these different cases:

1. At least one of $Q1, Q2$ is an axiom. Then we proceed as in point 1. of the proof of Theorem 1.
2. The cut-formula has the form $\neg{}^\circ F$ and the right cut formula also is the integral descendant of the principal formula of a $\neg - L4$ rule. Note that, by preliminary reductions and conventions, a right cut formula of the form $\neg{}^\circ F$ can be the integral descendant of $\neg - L4$ principal formulas or axioms only. As already remarked, $\neg - L4$ is a restriction of the classical $\neg - L2$, and the cut reduction is the same as in the classical case.
3. The cut-formula has the form ${}^\circ F$ and the left cut formula also is the integral descendant of the principal formula of a RCi - rule; we note that the right cut formula cannot be the principal formula of any logical rule. Then:

3.1 Let us consider the case where the left cut formula is the principal formula of a RCi - rule:

Q1

Q2

$$\frac{\frac{B \wedge \neg B, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \circ B} \quad \frac{\circ B, V \vdash Y}{\circ B, \Lambda \vdash X} \mathcal{R}}{\Gamma, \Lambda \vdash \Delta, X}$$

We examine the following sub-cases:

3.1.1 The rule \mathcal{R} on the right is not a $\neg - L3$ rule having the right cut-formula as constraint formula. Therefore, we produce a cut \mathcal{S} between the right \mathcal{C} -premise and the \mathcal{R} -premise, with $level(\mathcal{S}) < level(\mathcal{C})$, and apply the rule \mathcal{R} to the \mathcal{S} -conclusion.

3.1.2 The rule \mathcal{R} on the right is a $\neg - L3$ rule having the right cut-formula $\circ B$ as constraint formula:

Q1

Q2

$$\frac{\frac{B \wedge \neg B, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \circ B} \quad \frac{\circ B, V \vdash X, B}{\circ B, \neg B, V \vdash X}}{\neg B, \Gamma, V \vdash \Delta, X}$$

then we produce the following proof:

$$\frac{\frac{\frac{B \wedge \neg B, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \circ B} \quad \circ B, V \vdash X, B}{\Gamma, V \vdash \Delta, X, B} \quad B, \neg B, \Gamma \vdash \Delta}{\neg B, \Gamma, V \vdash \Delta, X} N$$

where the upper cut is reduced by induction on the level, the end-cut is reduced by induction on the rank. The right premise of the end-cut is obtained by Lemma 2 as the root of a cut-free **CI**-proof N from the proof $Q1$ of $B \wedge \neg B, \Gamma \vdash \Delta$.

3.2. The left cut formula is not the principal formula in the left cut-premise, and the rule \mathcal{R} on the right is not a $\neg - L3$ rule having the right cut-formula as constraint formula. Then let us consider for example the premise $Z \vdash W, \circ B$ of the rule \mathcal{G} in $Q1$ having the left cut premise $\Gamma \vdash \Delta, \circ B$ as conclusion, supposing that \mathcal{G} is a one premise rule. Then we produce the following proof:

$$\frac{\frac{Z \vdash W, \circ B \quad \circ B, \Lambda \vdash X}{Z, \Lambda \vdash W, X} \mathcal{G}}{\Gamma, \Lambda \vdash \Delta, X}$$

where the level of the introduced cut is lower than that of \mathcal{C} , and the cutrank is the same. If \mathcal{G} is a two-premise rule the reduction is similar.

3.3 The left cut-formula is not the principal formula in the left cut-premise, and the rule \mathcal{R} on the right is a $\neg - L3$ rule having the right cut-formula as constraint formula:

$$\frac{\begin{array}{c} Q1 \\ \frac{Z \vdash W, \circ B}{\Gamma \vdash \Delta, \circ B} \end{array} \quad \begin{array}{c} Q2 \\ \frac{\circ B, V \vdash X, B}{\circ B, \neg B, V \vdash X} \end{array}}{\neg B, \Gamma, V \vdash \Delta, X}$$

Let us consider the sub-proof $Q1$ of the left \mathcal{C} -premise $\Gamma \vdash \Delta, \circ B$ in P . We observe that, by hypotheses, the occurrence of $\circ B$ in such premise can be introduced, in the most general case, by a set \mathbf{E} of axiom occurrences of the form $\circ B \vdash \circ B$ and by a set \mathbf{H} of RCi -rule occurrences. Then, we replace each axiom $\circ B \vdash \circ B$ of \mathbf{E} with the proof of the \mathcal{C} -premise $\circ B, \neg B, V \vdash X$, and delete each RCi -rule of \mathbf{H} . Thus, we obtain, in the most general case, a cut free proof M of $B \wedge \neg B, \neg B, \Gamma, V \vdash \Delta, X$. Then, we produce the following proof:

$$\frac{\begin{array}{c} M \\ \frac{B \wedge \neg B, \neg B, \Gamma, V \vdash \Delta, X}{\neg B, \Gamma, V \vdash \Delta, X, \circ B} \end{array} \quad \begin{array}{c} Q2 \\ \frac{\circ B, V \vdash X, B}{\circ B, \neg B, V \vdash X} \end{array}}{\neg B, \Gamma, V \vdash \Delta, X}$$

where the introduced cut is included in the case 3.1.2. ■

As we can see, the RCi -rule has a remarkable deletion power. Thus, the subformula property cannot hold in the standard way for a cut-free \mathbf{CI} -proof. However, a *reduced subformula property* is possible.

Proposition 5 *Let A be a formula occurrence in a \mathbf{CI} -proof P . Then: i) if A has not the form $\neg B$ or $B \wedge \neg B$ then A occurs as subformula in the P -root. ii) If A has the form $\neg B$ or $B \wedge \neg B$, then either A occurs as subformula in the P -root or $\circ B$ occurs as subformula in the P -root.*

Proof. It is straightforward. ■

It must be stressed that \mathbf{CI} cannot define the connective $\circ(\cdot)$. In particular, the following sequents are \mathbf{CI} -provable: $\vdash \circ B \rightarrow \neg(B \wedge \neg B)$, $\vdash \neg \circ B \leftrightarrow B \wedge \neg B$. But $\vdash \neg(B \wedge \neg B) \rightarrow \circ B$ is not \mathbf{CI} -provable if $\circ B$ is not a \mathbf{CI} -theorem, as

by cut-elimination is evident. Moreover, the fact that in **Ci** the connectives \wedge and \vee in general do not commute if they occur in the scope of a \neg occurrence, becomes evident by properties of proofs in **CI**: in general, $\neg(B \wedge F) \vdash \neg(F \wedge B)$ (resp. $\neg(B \vee F) \vdash \neg(F \vee B)$) cannot be **CI**-provable for each pair $\langle B, F \rangle$ with B different from F , such that $\vdash \neg(F \wedge B)$ (resp. $\vdash \neg(F \vee B)$) is not **CI**-provable, as by cut-elimination is evident. Moreover, $B \wedge \neg B \vdash \neg B \wedge B$ and $B \vee \neg B \vdash \neg B \vee B$ are easily proved in **CI** as in the classical **LK** through positive rules, but neither (i) $\neg(B \wedge \neg B) \vdash \neg(\neg B \wedge B)$ nor (ii) $\neg(B \vee \neg B) \vdash \neg(\neg B \vee B)$ are *in general* **CI**-provable.

Example 1 The following are cut-free proofs in **CI** of some relevant **Ci**-theorems:

$$\frac{\frac{\frac{\circ A \vdash \circ A}{\circ A, \neg \circ A \vdash} \neg - L4}{\circ A \wedge \neg \circ A \vdash} \wedge -L}{\vdash \circ \circ A} RCi$$

$$\frac{\frac{\frac{\frac{A \vdash A}{\circ A, A \vdash A} W - R}{\circ A, \neg A, A \vdash} \neg - L3}{\circ A, \neg A, \neg \neg A \vdash} \neg - L1}{\frac{\frac{\circ A, \neg A \wedge \neg \neg A \vdash}{\circ A \vdash \circ \neg A} \wedge -L}{\vdash \circ A \rightarrow \circ \neg A} RCi} \rightarrow -R$$

$$\frac{\frac{\frac{A \vdash A}{A \wedge \neg A \vdash A} \wedge -L}{\vdash A, \circ A} RCi}{\vdash A \vee \circ A} \vee -R$$

■

As expected, the set of **CI** bottom particles properly extends the set of **BC** bottom particles. For example, each formula of the form $\neg \circ F$ is a bottom particle of **CI** but not of **BC**.

5 The system **CIL**

Cil is given by **Ci** plus $\neg(A \wedge \neg A) \rightarrow \circ A$. As in [8] is explained, the **C**-system **Cil** is close to the system **C**₁ of Da Costa [12]³. In **Cil** the connective $\circ(\cdot)$ remains as primitive in the language, but the equivalence $\neg(B \wedge \neg B) \leftrightarrow \circ B$ is a **Cil**-theorem: thus **Cil** defines $\circ(\cdot)$ through the remaining standard connectives. As a consequence, differently from **bC** and **Ci**, **Cil** may be trivialized by classical formulas too. We define the sequent system **CIL** as follows:

$$\mathbf{CI} \quad \text{plus} \quad \frac{\Gamma \vdash \Delta, A \wedge \neg A}{\neg(A \wedge \neg A), \Gamma \vdash \Delta} \quad \neg - L5$$

The rule $\neg - L5$ is the *proper CIL*-rule. In the rule $\neg - L5$ the formula $A \wedge \neg A$ in the premise succedent is the $\neg - L5$ *auxiliary formula*, the formula $\neg(A \wedge \neg A)$ in the conclusion antecedent is the $\neg - L5$ *principal formula*.

Proposition 6 **CIL** is a sequent version of the system **Cil** presented in [9].

Proof. We produce the following *canonical CIL*-proof $Q_{\mathbf{Cil}}$ of the **Cil**-axiom $\neg(A \wedge \neg A) \rightarrow \circ A$:

$$\frac{\frac{\frac{A \vdash A \quad \neg A \vdash \neg A}{A, \neg A \vdash A \wedge \neg A} \quad \wedge -R}{\neg(A \wedge \neg A), \neg A, A \vdash} \quad \neg - L5}{\frac{\frac{\neg(A \wedge \neg A), A \wedge \neg A \vdash}{\neg(A \wedge \neg A), A \wedge \neg A \vdash} \quad \wedge -L}{\neg(A \wedge \neg A) \vdash \circ A} \quad RCi}$$

On the other hand, **CI** plus $\neg(A \wedge \neg A) \vdash \circ A$ has the same theorems as **CIL**.

In fact, assuming $X \vdash Y, A \wedge \neg A$, and recalling that $A \wedge \neg A \vdash A$ is an obvious **BC**-theorem, we have:

$$\frac{\frac{\frac{X \vdash Y, A \wedge \neg A \quad A \wedge \neg A \vdash A}{X \vdash Y, A} \quad Cut}{\circ A, X \vdash Y, A} \quad W - L}{\circ A, \neg A, X \vdash Y} \quad \neg - L3$$

we employ the conclusion of the derivation above as right premise of the following:

³ **Cil** can be seen as the system **C**₁ of Da Costa [12] with the exclusion of the so called *consistency propagation axioms* (see [8] Sec. 5.2, Def. 108)

$$\begin{array}{c}
\frac{\frac{\frac{\neg(A \wedge \neg A) \vdash \circ A \quad \circ A, \neg A, X \vdash Y}{\neg(A \wedge \neg A), \neg A, X \vdash Y} \text{Cut}}{\neg(A \wedge \neg A), A \wedge \neg A, X \vdash Y} \wedge -L}{(A \wedge \neg A) \wedge \neg(A \wedge \neg A), X \vdash Y} \wedge -L \\
\hline
X \vdash Y, \circ(A \wedge \neg A) \quad \text{RCi}
\end{array}$$

where the left premise is the **Cil**-axiom. On the other hand, we also have:

$$\frac{\frac{X \vdash Y, A \wedge \neg A}{\circ(A \wedge \neg A), X \vdash Y, A \wedge \neg A} W - L}{\circ(A \wedge \neg A), \neg(A \wedge \neg A), X \vdash Y} \neg - L3$$

from which, by cut with the sequent $X \vdash Y, \circ(A \wedge \neg A)$ obtained above, we get $\neg(A \wedge \neg A), X \vdash Y$. ■

It is worth noting that, as expected, **CIL** has infinitely many *classical* bottom particles of arbitrary complexity. In particular, for each **CI** bottom particle F having a conjunct of the form $\circ B$, the replacement of the conjunct $\circ B$ with $\neg(B \wedge \neg B)$ gives a **CIL** bottom particle: for example $A \wedge \neg A \wedge \neg(A \wedge \neg A)$ is a **CIL** bottom particle for each A , and, if A is classical, it is not a **CI**-bottom particle. If A has the form $\circ B$, then it is a **CI** bottom particle too.

In order to prove the cut elimination theorem for **CIL**, we employ the following strategy: the goal is to show that any **CIL**-proof P can be transformed into a **CIL**-proof Q with the same root, such that no cut formula in Q is introduced by $\neg - L5$ rule occurrences.

Lemma 4 *Let P be a cut free proof in **CIL** of the sequent $S \equiv X \vdash Y, A \wedge \neg A$. Then, we can obtain from P a cut free **CIL**-proof $P1$ of $X \vdash Y, A$ and a cut free **CIL**-proof $P2$ of $X \vdash Y, \neg A$.*

Proof. In the most general case the direct ancestors of the occurrence of $A \wedge \neg A$ in S are introduced in P by a set **H** of right weakenings, by a set **M** $\equiv \{(A \wedge \neg A \vdash A \wedge \neg A)_i\}$ of logical axiom occurrences and by a set **J** $\equiv \left\{ \frac{W_j \vdash Z_j, A \quad V_j \vdash U_j, \neg A}{W_j, V_j \vdash Z_j, U_j, A \wedge \neg A} \right\}$ of $\wedge - R$ rule occurrences. We produce in P the following reductions: the elements of **H** are replaced by weakenings that introduce A ; each element of **M** is replaced by the following proof: $\frac{A \vdash A}{A \wedge \neg A \vdash A}$; each rule occurrence of **J** is replaced by the proof of the left premise $W_j \vdash Z_j, A$. Thus, by adding also possible suitable weakenings, we obtain a proof $P1$ of $X \vdash Y, A$. The construction of $P2$ is symmetrical. ■

Lemma 5 *Let P be a cut free **CIL**-proof of the sequent $S \equiv \neg(A \wedge \neg A), X \vdash Y$. If all direct ancestors of $\neg(A \wedge \neg A)$ are introduced by axioms or weakenings only, we can obtain from P a cut free proof P' of the sequent $\circ A, X \vdash Y$. If*

the direct ancestors of $\neg(A \wedge \neg A)$ in P are also introduced by $\neg - L5$ rule occurrences, then we can obtain from P a cut free proof Q of the sequent ${}^\circ A, \neg A, X \vdash Y$, where no direct ancestor of $\neg A$ is introduced by $\neg - L5$ rules.

Proof. First we produce the following canonical cut-free **CIL**-proof of ${}^\circ A \vdash \neg(A \wedge \neg A)$:

$$\frac{\frac{\frac{A \vdash A}{{}^\circ A, A \vdash A} W - L}{{}^\circ A, \neg A, A \vdash} \neg - L3}{\frac{{}^\circ A, A \wedge \neg A \vdash}{{}^\circ A \vdash \neg(A \wedge \neg A)} \wedge - L} \neg - R$$

Let $\mathbf{H} \equiv \{(\neg(A \wedge \neg A) \vdash \neg(A \wedge \neg A))_i\}$ be the set of axiom occurrences introducing in P a direct ancestor of the formula $\neg(A \wedge \neg A)$ of S , and let \mathbf{J} be the set of left weakenings introducing in P a direct ancestor of the formula $\neg(A \wedge \neg A)$ of S . We replace each element of \mathbf{H} with the canonical proof of ${}^\circ A \vdash \neg(A \wedge \neg A)$ and each element of \mathbf{J} with a left weakening introducing ${}^\circ A$, getting a proof P' of ${}^\circ A, X \vdash Y$. In the most general case, suppose also that $\mathbf{V} \equiv \left\{ \frac{W_j \vdash Z_j, A \wedge \neg A}{\neg(A \wedge \neg A), W_j \vdash Z_j} \right\}$ is the set of $\neg - L5$ rule occurrences introducing in P a direct ancestor of the formula $\neg(A \wedge \neg A)$ of S . We replace each element of \mathbf{V} with the following cut free proof

$$\frac{\frac{W_j \vdash Z_j, A}{{}^\circ A, W_j \vdash Z_j, A} W - L}{{}^\circ A, \neg A, W_j \vdash Z_j} \neg - L3$$

where the uppermost sequent is given by Lemma 4, getting a cut free proof Q of the sequent ${}^\circ A, \neg A, X \vdash Y$. ■

Lemma 6 *Let P be a cut free **CIL**-proof of the sequent $S \equiv X \vdash Y, \neg(A \wedge \neg A)$. Then we can obtain from P a cut free proof Q of the sequent $X \vdash Y, {}^\circ A$.*

Proof. In the most general case the direct ancestors of the occurrence of $\neg(A \wedge \neg A)$ in S are introduced in P by a set \mathbf{H} of right weakenings, by a set $\mathbf{M} \equiv \{(\neg(A \wedge \neg A) \vdash \neg(A \wedge \neg A))_i\}$ of logical axiom occurrences and by a set $\mathbf{J} \equiv \left\{ \frac{A \wedge \neg A, W_j \vdash Z_j}{W_j \vdash Z_j, \neg(A \wedge \neg A)} \right\}$ of $\neg - R$ rule occurrences. We produce in P the following reductions: each weakening of \mathbf{H} is replaced by a weakening introducing ${}^\circ A$; each element of \mathbf{M} is replaced by the canonical proof Q_{CIL} of $\neg(A \wedge \neg A) \vdash {}^\circ A$ (Proposition 6); each rule occurrence of \mathbf{J} is replaced by the *RCi* instance: $\frac{A \wedge \neg A, W_j \vdash Z_j}{W_j \vdash Z_j, {}^\circ A}$. Thus, we obtain a proof Q of $X \vdash Y, {}^\circ A$. ■

Lemma 7 *Let Q be a **CIL**-proof having a cut \mathcal{C} with cut formula of the form $\neg(A \wedge \neg A)$ as end rule, such that the sub-proofs of the \mathcal{C} -premises are cut free:*

$$\frac{\begin{array}{c} Q1 \\ \Gamma \vdash \Delta, \neg(A \wedge \neg A) \end{array} \quad \begin{array}{c} Q2 \\ \neg(A \wedge \neg A), \Lambda \vdash X \end{array}}{\Gamma, \Lambda \vdash \Delta, X}$$

Then, we can replace Q with a proof H of the following form:

$$\frac{\begin{array}{c} H1 \\ \Gamma \vdash \Delta, \circ A \end{array} \quad \begin{array}{c} H2 \\ \circ A, \Lambda \vdash X \end{array}}{\Gamma, \Lambda \vdash \Delta, X}$$

if the right cut formula of \mathcal{C} has no direct ancestor introduced by a $\neg - L5$ rule occurrence, or, in the most general case, with a proof M of the following form:

$$\frac{\begin{array}{c} M1 \\ \Gamma \vdash \Delta, \circ A \end{array} \quad \begin{array}{c} M2 \\ \circ A, \neg A, \Lambda \vdash X \end{array}}{\begin{array}{c} M3 \\ \Lambda \vdash X, \neg A \end{array}} \quad \frac{\Lambda \vdash X, \neg A \quad \neg A, \Gamma, \Lambda \vdash \Delta, X}{\Gamma, \Lambda \vdash \Delta, X}$$

where the right cut formula of the lower cut is not introduced in M by $\neg - L5$ rule occurrences.

Proof. The construction of H , i.e. of the cut free sub-proofs $Q1$ and $Q2$ follows immediately from Lemmas 5 and 6. In the construction of M , the cut free sub-proofs $M1$ and $M2$ follows immediately from Lemmas 5 and 6. For the cut free proof $M3$ of the sequent $\Lambda \vdash X, \neg A$, let us consider the set $\mathbf{V} \equiv \left\{ \frac{W_j \vdash Z_j, A \wedge \neg A}{\neg(A \wedge \neg A), W_j \vdash Z_j} \right\}$ of $\neg - L5$ rule occurrences introducing in Q a direct ancestor of the right cut formula $\neg(A \wedge \neg A)$ of \mathcal{C} . By Lemma 4, for each premise $W_j \vdash Z_j, A \wedge \neg A$ in \mathbf{V} we can get a cut free proof N_j of $W_j \vdash Z_j, \neg A$. If we replace in $Q2$ each element of \mathbf{V} with N_j , we get a cut free proof of $\Lambda \vdash X, \neg A$. ■

Corollary 1 *Let P be a proof in **CIL**. Then we can obtain from P a **CIL**-proof Q where in each uppermost cut no cut formula is introduced by a $\neg - L5$ rule. ■*

Lemma 8 *Let P be any proof in **CIL** of the sequent $X \vdash Y$. Then, a suitable*

PRA-formulation \mathcal{D} of the cut reductions prescribed by the cut-elimination procedure for **CI** of cuts whose right cut formula is not the integral descendant of the principal formula of any $\neg - L5$ rule exists, such that \mathcal{D} can be applied to P , obtaining a cut free **CIL**-proof P' of $X \vdash Y$.

Proof. Recalling the proofs of the analogous Lemmas 1 and 3 for **BC** and **CI**, in the mentioned cases the procedure \mathcal{D} do not break any constraint. ■

Theorem 3 *The system **CIL** admits cut-elimination.*

Proof. By Corollary 1 we can consider only proof-trees P in **CIL** such that in any uppermost cut no right cut formula is introduced by $\neg - L5$ rule occurrences. Then, the thesis follows from Lemma 8. ■

CIL too has the *reduced subformula property*.

Proposition 7 *Let A be a formula occurrence in a cut free **CIL**-proof P . Then: i) if A has not the form $\neg B$ or $B \wedge \neg B$ then A occurs as subformula in the P -root. ii) If A has the form $\neg B$ or $B \wedge \neg B$, then either A occurs as subformula in the P -root or $\circ B$ occurs as subformula in the P -root. ■*

The proof is the same as for **CI**.

Here are some *simple* cut free proofs of significant **CIL**-theorems.

Consider $S \equiv \neg(A \wedge \neg A) \vdash \neg(\neg A \wedge A)$, asserting a partial commutation of \wedge inside the scope of \neg for any arbitrary A :

$$\frac{\frac{\frac{A \vdash A \quad \neg A \vdash \neg A}{A, \neg A \vdash A \wedge \neg A} \wedge -R}{\neg(A \wedge \neg A), \neg A, A \vdash} \neg - L5}{\frac{\neg(A \wedge \neg A), \neg A \wedge A \vdash}{\neg(A \wedge \neg A) \vdash \neg(\neg A \wedge A)} \wedge -L} \neg - R$$

On the other hand, by cut elimination is evident that the converse $\neg(\neg A \wedge A) \vdash \neg(A \wedge \neg A)$ is not, *in general*, a **CIL**-theorem, since $\neg A \wedge A$ cannot be the auxiliary formula of a $\neg - L5$ rule. Moreover, S is not, *in general*, a **CI**-theorem (Section 4). Thus, as expected, the set of **CIL**-theorems in classical language properly extends the set of **CI**-theorems in classical language. This is already evident, by cut elimination, for the conclusion of any simplest instance of the $\neg - L5$ rule: the sequent $\neg(p \wedge \neg p), \neg p, p \vdash$, with p atom, *is not* a **CI**-theorem. The following is a cut free proof of the relevant **CIL**-theorem $\circ(A \wedge \neg A)$:

$$\begin{array}{c}
\frac{A \vdash A \quad \neg A \vdash \neg A}{A, \neg A \vdash A \wedge \neg A} \quad \wedge -R \\
\frac{\quad}{\neg(A \wedge \neg A), \neg A, A \vdash} \quad \neg - L5 \\
\frac{\quad}{\neg(A \wedge \neg A), A \wedge \neg A \vdash} \quad \wedge -L \\
\frac{\quad}{(A \wedge \neg A) \wedge \neg(A \wedge \neg A) \vdash} \quad \wedge -L \\
\hline
\vdash \circ(A \wedge \neg A) \quad RCi
\end{array}$$

Observe that neither $(A \wedge \neg A) \wedge \neg(A \wedge \neg A)$ nor $\neg(A \wedge \neg A)$ occur as subformulas in the root, but the reduced subformula property (Proposition 7) is respected, since $\circ(A \wedge \neg A)$ occurs in the root. Moreover, by cut elimination we see that $\vdash \circ(A \wedge \neg A)$ cannot be the root of any **CI**-proof for any *classical* formula A . Then, by $\neg - L4$, we see that, for any classical A , $\neg \circ(A \wedge \neg A)$ is a *non-classical* **CIL** bottom particle which is not a **CI** bottom particle. From the *RCi*-premise of the proof above, by $\neg - R$, we have $\vdash \neg((A \wedge \neg A) \wedge \neg(A \wedge \neg A))$ as **CIL**-theorem for each A : thus, if A ranges over *classical* formulas, **CIL** proves infinitely many classical instances of the non-contradiction principle.

6 A notion of constructivity for paraconsistency: CI as a constructive logic

6.1 Heuristic preliminaries and suggestions from the literature

Let us suppose to examine a sequent system **W** whose theorems are included in that of classical logic **LK**, and whose rules are obtained by weakening or constraining some **LK**-rules. Let's suppose to ask the following question, from an heuristic standpoint, forgetting the thousands of papers that have been written about constructive logic in the last century: what are the minimal properties that **W** must have to be a "constructive" logical system? Our intuition would propose such necessary conditions: i) some negation rules of **W** are constrained w.r.t the negation rule of **LK**; ii) **W** admits cut-elimination. The more relevant condition is i): it reflects our seminal insight that, first of all, in a constructive logic, negation cannot be used as in the classical setting. As to ii), it is quite natural: we remark that, in general, each *effective* proof-search strategy for the **W**-proof of a given sequent S requires the cut elimination property. After this, if we also consider the historical development of the idea of constructive logic, we must observe that it is strictly linked to intuitionistic logic and its subsystems. Moreover, intuitionistic logic respects the above mentioned conditions: **LJ** has the cut elimination property, and has a constrained right negation rule:

$$\frac{A, \Gamma \vdash}{\Gamma \vdash \neg A} \neg - R$$

where the premise succedent must be empty. Note that this is a specific proper constraint, qualifying intuitionistic logic. Indeed, even if the easy formulation of **LJ** says that the rules are such that each end sequent in a proof has at most a singleton as succedent, in the Maehara sequent version of intuitionistic logic (see [23] p. 52) the only constraint is that the *critical* rules $\neg - R$, $\rightarrow - R$, $\forall - R$ are allowed only if the principal formula is the only formula in the conclusion succedent. However, though important historical justifications could support it, *we reject the identification between intuitionistic logic and constructive logic*. That is, we think that different approaches to constructivism in logic can exist: a first one can emphasize the criticism to the excluded middle principle, while a second one could develop the criticism to the non-contradiction principle. We deem that *today* (and not one century ago) both positions have relevant mathematical motivations. *Thus, we are searching for a satisfactory characterization of a class of paraconsistent logics as constructive logics*. Obviously, the mere paraconsistency without further conditions cannot give constructivity: in our opinion only a very peculiar class of paraconsistent logics could be said *constructive*.

Preliminarily, though rejecting identification, the strong connection between intuitionistic logic and constructivism must be considered. That is, we assume that *in principle*, a paraconsistent constructive logic **U** should present a *canonical link* with an intuitionistic logic **V**, that we call the *reference system for U*, such that, *essentially*, the negation rules of the sequent version of **U** can be obtained through suitable canonical transformations from the negation rules of the sequent version of **V**. This is a heuristic ideal requirement that will be made technical in the sequel. In order to get some suggestions for the definition of such canonical link, it can be useful to examine the well studied *duality relations* between intuitionistic and paraconsistent logic. Note however that we will maintain as separate notions the canonical link between any paraconsistent system **U** and its possible intuitionistic reference system **V** (for which we are searching a suitable formal definition), and the duality relation between intuitionistic and paraconsistent logic (which is a canonical topic).

A first kind of duality relation studied in the literature is based on the role of logical connectives; we call it *connective-duality relation* and it is studied, for example, in Brunner-Carnielli [5]. In [5] *multiple deductive systems* $S \equiv (L, \vdash_S)$ are considered where L is a propositional language and \vdash_S is a *consequence relation* such that, if Γ and Δ are sets of L -formulas, the intended meaning of $\Gamma \vdash_S \Delta$ is that, assuming the conjunction of the propositions of Γ , at least one element of Δ holds; moreover, $\Gamma \vdash_S \Delta$ is closed under the usual structural rules (weakening, cut) of sequent calculus, once the identification between $\Gamma \vdash_S \Delta$ and the sequent $\Gamma \vdash \Delta$ is accepted, and a deduction theorem for S holds. The dualization operation $(.)^*$ gives the dual connectives of the standard propositional ones in this way: $\perp^* \equiv \top$, $(A \wedge B)^* \equiv A^* \vee B^*$, $(A \vee$

$B)^* \equiv A^* \wedge B^*$, $(A \rightarrow B)^* \equiv A^* - B$, $(\neg A)^* \equiv \top - A^*$ where “ $-$ ” is the (intensional) *pseudo-difference* or *coimplication* connective, such that $D - E$ has the heuristic meaning “ D but not E ” (see also [25], [1]). A *dual consequence relation* \vdash_{S^*} can be defined, such that the *dual deductive system* $S^* \equiv (L^*, \vdash_{S^*})$ has the following property: $\Gamma^* \vdash_{S^*} \Delta^*$ if and only if $\Delta \vdash_S \Gamma$. Moreover, the (connective-)dual logic of an intuitionistic logic is always a paraconsistent logic ([5] p. 169).

What suggestions arise from the results presented in Brunner-Carnielli [5] on the *connective-duality* between intuitionism and paraconsistency? If we see the consequence statement $\Delta \vdash_S \Gamma$ as the end sequent $\Delta \vdash \Gamma$ of a proof in a sequent formulation of S (assuming that a deduction theorem holds) the connective-duality is expressed by an exchange antecedent/succedent (i.e. left/right) in the sequents and by a change standard/dual in the connectives: if $\Gamma \vdash \Delta$ is the root of a proof in the intuitionistic system S , then $\Delta^* \vdash \Gamma^*$ is the root of the dual proof in the paraconsistent dual S^* . Therefore, we can accept the following suggestion: *the canonical link between a paraconsistent constructive logic \mathbf{U} and its reference system \mathbf{V} should express a kind of antisymmetry between the negation rules of the sequent versions of \mathbf{U} and \mathbf{V} .*

A second kind of duality relation studied in the literature, is directly defined starting from the sequent version of intuitionistic logic: we call it *sequent-duality relation*. In this way, for example, the paraconsistent systems **LDJ** of Urbas [25] and **DI** of Aoyama [1] are defined. Since the two systems are similar, we will focus on **LDJ**. **LDJ** can be obtained from the standard intuitionistic sequent system **LJ**, where each sequent in a proof has at most a singleton as succedent, through the converse condition: in **LDJ** each sequent in a proof has at most a singleton as antecedent. **LDJ** is called a *dual-intuitionistic* sequent system ([25] p. 440). Let’s consider the negation rules of **LDJ**:

$$\frac{\vdash \Delta, A}{\neg A \vdash \Delta} \neg - L \qquad \frac{A \vdash \Delta}{\vdash \Delta, \neg A} \neg - R$$

the proper constraint is in $\neg - L$: the negated formula can be introduced in the conclusion antecedent only if the premise antecedent is empty. Remark that $\neg - L$ of **LDJ** can be obtained from $\neg - R$ of **LJ** by exchanging antecedent and succedent in the sequents, i.e. left side and right side. The systems presented in [25] and [1] give an important contribution to the investigation of sequent-duality between paraconsistency and intuitionism. The relevance of **LDJ** is in the exemplification of the duality, and it is not diminished by the below considerations on the intrinsic proof-theoretic features of **LDJ**, which are an independent topic.

Thus, sequent-duality confirms what we already observed in defining the sequent systems **BC**, **CI**, **CIL**: in general, negation rules of paraconsistent systems must have constraints on the left side of the premise. However, we

note that paraconsistent systems obtained from intuitionistic logic only by a left/right exchange in the sequents, *do not offer an expressive and informative paraconsistency*. **LDJ** does not prove the formalized modus ponens $A \wedge (A \rightarrow B) \vdash B$, so that positive logic loses relevant fragments without any substantial motivation. On the other hand, for each **LDJ**-theorem $\vdash A$, the sequent $A \wedge \neg A \vdash$ is provable too, and this is a weak form of paraconsistency. As to a possible *constructivity*, the fact that to each theorem B a contradiction $B \wedge \neg B$ corresponds that trivializes **LDJ**, independently of the information included in B , does not seem a *constructive* feature for a paraconsistent system.

From the examined sequent-duality relation we can derive two types of suggestions. The first one confirms that between the sequents of the negation rules of any paraconsistent constructive system **U** and those of its intuitionistic reference system **V**, a kind of antisymmetry relation should hold. The second one is a negative reflexion: the empty antecedent condition for the $\neg - L$ rule is a little informative constraint, giving a poor paraconsistency, which is in addition not much *constructive*, assuming that *constructive* implies *effectively informative*. The same does not hold for the empty succedent condition in the $\neg - R$ rule of intuitionistic logic **LJ**, and this is not odd, since intuitionism has completely different inferential requirements. Thus, taking into account the definition of negation rules for **BC**, **CI**, **CIL** already presented in the previous sections, we generalize the notion of *constraint formula* for the $\neg - L$ rules of a paraconsistent sequent system **U**: in general, we say that **U** has the *constraint formula property* if at least one of the $\neg - L$ rules of **U** has the form:

$$\frac{\delta, X \vdash \Delta, A}{\neg A, \delta, X \vdash \Delta}$$

where the formula δ is neither a theorem nor a bottom particle of **U** and is the *constraint formula of the rule*. δ is effectively informative: it declaratively expresses the *distance* between the system **U** and classical logic **LK**. The case of the empty antecedent constraint can be seen as the limit case of the *trivial constraint formula*.

6.2 Formal definition of constructive paraconsistency

We now proceed to formalize the heuristic considerations presented above. We assume to work always with sequent formulated system. We have said that, in order to define a general notion of constructive logic, we focus on the negation rules of the system. However, in our opinion, the notion of constructivity is linked not only to the necessity of constraining classical negation, but also to the inclusion in the system of adequately powerful negation rules. Only systems with a sufficiently expressive negation allow interesting constructive

inference. Thus, any positive fragment of classical logic **LK** could be said *constructive* only in the sense of *trivially constructive*. But we think that, for example, also the system given by **positive(LK)** plus $\frac{A, \Gamma \vdash \Delta}{\neg\neg A, \Gamma \vdash \Delta} \neg\neg L1$ cannot be qualified as a *constructive paraconsistent* system since $\neg\neg L1$ allows a too weak use of negation. Then, we will discuss about constructivity only for paraconsistent systems endowed by *diagonal* negation rules:

Definition 3 *A sequent rule introducing negation is diagonal if the auxiliary formula is at the left (right) side of the premise and the principal formula is at the right (left) side of the conclusion. We call left (right) diagonal negation rules those with the principal formula at the left (right) side of the conclusion.*

■

We will focus on *diagonal* negation rules. The notion is clear: a diagonal negation rule not only introduces one or more occurrences of \neg , but also *moves* the formula, changing the sequent side at which it occurs. A relevant class of diagonal negation rules is given by rules with the *constraint formula property*:

Definition 4 *i) A left diagonal negation rule of a system **U** has the constraint formula property if it has the following form:*

$$\frac{\Phi, \Gamma \vdash \Delta, A}{\neg A, \Phi, \Gamma \vdash \Delta} \mathcal{H}$$

where exactly one of the following cases holds:

- a) the set Φ is empty, and A is a fixed formula schema Λ ;
- b) no constraints are posed on A and the set Φ is a singleton including a fixed formula schema Σ , such that infinitely many instances of Σ are neither theorems nor bottom particles of **U**. In each \mathcal{H} -instance, the Σ -instance δ is called the *constraint formula of the rule instance*.

ii) *A right diagonal negation rule of a system **U** has the constraint formula property if it has the following form:*

$$\frac{B, \Gamma \vdash \Delta, \Psi}{\Gamma \vdash \Delta, \Psi, \neg B} \mathcal{R}$$

where exactly one of the following cases holds:

- c) the set Ψ is empty and B is a fixed formula schema Θ ;
- d) no constraints are posed on B and the set Ψ is a singleton including a fixed formula schema Π , such that infinitely many instances of Π are neither theorems nor bottom particles of **U**. In each \mathcal{H} -instance, the Π -instance σ is

called the constraint formula of the rule instance. ■

Remark 1 The previous definition needs some comments. It is based on one assumption, which is supported by some examples and by the suggestions given by the paraconsistency/intuitionism dualities. The assumption is the following: we deem that paraconsistent systems are characterized by left diagonal negation rules having proper constraints on the left side (for example the $\neg - L3$ rule) and that intuitionistic systems are characterized by right diagonal negation rules having proper constraints on the right side (for example the empty premise succedent condition in $\neg - R$ of **LJ**), with the specification that, in some cases, the constraint is given by a fixed formula schema for the auxiliary formula: in such cases, the constraint imposes a particular form of the principal formula of the rule, i.e. at the left side of the conclusion in paraconsistent systems, and at the right side of the conclusion in (pseudo)-intuitionistic systems. For these reasons we establish that it is relevant to formalize left diagonal rules constrained on the left, and right diagonal rules constrained on the right. **BC**, **CI**, **CIL** provide examples of left diagonal rules with the constraint formula property:

$$\frac{\circ A, \Gamma \vdash \Delta, A}{\circ A, \neg A, \Gamma \vdash \Delta} \neg - L3 \quad \frac{\Gamma \vdash \Delta, \circ A}{\neg \circ A, \Gamma \vdash \Delta} \neg - L4 \quad \frac{\Gamma \vdash \Delta, A \wedge \neg A}{\neg(A \wedge \neg A), \Gamma \vdash \Delta} \neg - L5$$

$\neg - L3$ is in the case b) of the definition and the schema Σ is $\circ A$. $\neg - L4$ and $\neg - L5$ are in the case a) of the definition, where the constraint is conveyed by the auxiliary formula. The schemas Λ are respectively $\circ A$ and $A \wedge \neg A$. Note that all these systems has the right diagonal negation rule $\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \neg - R$ which has not the constraint formula property. The following right diagonal negation rule: $\frac{\bullet A, \Omega \vdash \Pi}{\Omega \vdash \Pi, \neg \bullet A} \neg - R4$, where $\bullet(.)$ is the black circle connective of Logic of Formal Inconsistency, has the constraint formula property with the constraint conveyed by the fixed schema $\bullet A$ of the auxiliary formula, so that the principal formula, at the right side of the conclusion, has the particular form $\neg \bullet A$. ■

At this point, recalling what already sketched in Section 6.1, we can produce a formal counterpart of the idea that a constructive paraconsistent system **U** should have an intuitionistic *reference system* **V** from which the negation rules of **U** can be obtained through canonical manipulations. Essentially, we can expect a kind of antisymmetry relation between premises and conclusions of the diagonal negation rules of the two systems, linking left diagonal negations of **U** to right diagonal negations of **V**. This imposes to enlarge the class of intuitionistic systems to that of *pseudo-intuitionistic* systems, defined below. The technical reason is simple: the standard right diagonal negation of intuitionistic systems, with the empty premise succedent condition, does not allow to get, by antisymmetry, interesting left diagonal negation rules, i.e.

interesting paraconsistent systems.

Definition 5 *Let \mathbf{V} be a sequent system in the language of Logic of Formal Inconsistency, including the positive rules of classical logic \mathbf{LK} and endowed by diagonal negation rules. Then \mathbf{V} is a pseudo-intuitionistic system if it has cut-elimination and, moreover, infinitely many instances of the excluded middle principle $B \vee \neg B$ and of the left double negation principle $\neg\neg B \rightarrow B$ are not \mathbf{V} -provable. ■*

Note that intuitionistic logic \mathbf{LJ} itself does not reject all the instances of the mentioned principles: it proves infinitely many excluded middle instances and left double negation instances (from each \mathbf{LJ} -provable $B \vdash$, the sequents $\vdash B \vee \neg B$, $\neg\neg B \vdash B$ are provable, and from each \mathbf{LJ} -provable $\vdash A$ the sequents $\vdash A \vee \neg A$, $\neg\neg A \vdash A$ are provable). Moreover, \mathbf{LFI} 's language includes classical language, so that Definition 5 obviously concerns also the pseudo-intuitionistic systems in classical language.

Then, a pseudo-intuitionistic system may have right diagonal negation rules which are constrained in many various ways, and not only by the empty premise succedent condition. Moreover, *constraint formula properties are possible.*

Definition 6 *Let \mathcal{H} be a left diagonal negation rule of a sequent system \mathbf{U} in the language of Logic of Formal Inconsistency. Then the negation-dual rule \mathcal{E} of \mathcal{H} is a right diagonal negation rule so defined:*

i) if \mathcal{H} has not the constraint formula property, i.e it has the form:

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \mathcal{H}$$

then \mathcal{E} has the form

$$\frac{B, \Omega \vdash \Pi}{\Omega \vdash \Pi, \neg B} \mathcal{E}$$

ii) if \mathcal{H} has the constraint formula property, i.e it has the form:

$$\frac{\Phi, \Gamma \vdash \Delta, A}{\neg A, \Phi, \Gamma \vdash \Delta} \mathcal{H}$$

where Φ and A have the properties established by Definition 4, then:

a) if Φ is empty and A is the fixed formula schema Λ then \mathcal{E} has the form

$$\frac{\neg \Lambda, \Omega \vdash \Pi}{\Omega \vdash \Pi, \neg\neg \Lambda} \mathcal{E}$$

b) if Φ is not empty then \mathcal{E} has the form:

$$\frac{B, \Omega \vdash \Pi, \neg\Sigma}{\Omega \vdash \Pi, \neg\Sigma, \neg B} \mathcal{E}$$

where Σ is the formula schema included in the singleton Φ . ■

Therefore: when the rule \mathcal{H} has not the constraint formula property its negation-dual \mathcal{E} is obvious. The interesting case arises when \mathcal{H} has the constraint formula property: in such case the constraint formula schema of \mathcal{E} is obtained by changing side and adding a negation connective \neg to the constraint formula schema of \mathcal{H} (that may coincide with the auxiliary formula, as the case a) above shows). Thus, we apply an antisymmetry operation to the constraint formula of \mathcal{H} . As evident, \mathcal{E} is a right diagonal negation rule with the constraint formula property.

Definition 7 Let \mathcal{G} be a right diagonal negation rule of a sequent system \mathbf{V} in the language of Logic of Formal Inconsistency. Then the negation-dual rule \mathcal{L} of \mathcal{G} is a left diagonal negation rule so defined:

i) if \mathcal{G} has not the constraint formula property, i.e it has the form:

$$\frac{B, X \vdash Y}{X \vdash Y, \neg B} \mathcal{G}$$

then \mathcal{L} has the form

$$\frac{\Gamma \vdash \Delta, F}{\neg F, \Gamma \vdash \Delta} \mathcal{L}$$

ii) if \mathcal{G} has the constraint formula property, i.e it has the form:

$$\frac{B, X \vdash Y, \Psi}{X \vdash Y, \Psi, \neg B} \mathcal{G}$$

where Ψ and B have the properties established by Definition 4, then:

a) if Ψ is empty and B is the fixed formula schema Θ then \mathcal{L} has the form:

$$\frac{\Gamma \vdash \Delta, \neg\Theta}{\neg\neg\Theta, \Gamma \vdash \Delta} \mathcal{L}$$

b) if Ψ is not empty then \mathcal{L} has the form:

$$\frac{\neg\Sigma, \Gamma \vdash \Delta, F}{\neg\Sigma, \neg F, \Gamma \vdash \Delta} \mathcal{L}$$

where Σ is the formula schema included in the singleton Ψ . ■

Proposition 8 *The negation-dual rules of the rules $\neg - L3$, $\neg - L4$, $\neg - L5$ of the systems **BC**, **CI**, **CIL** are the following:*

$$\frac{A, \Omega \vdash \Pi, \neg^\circ A}{\Omega \vdash \Pi, \neg^\circ A, \neg A} \neg - R3 \quad \frac{\neg^\circ A, \Omega \vdash \Pi}{\Omega \vdash \Pi, \neg^\circ \neg A} \neg - R4 \quad \frac{\neg(A \wedge \neg A), \Omega \vdash \Pi}{\Omega \vdash \Pi, \neg \neg(A \wedge \neg A)} \neg - R5$$

■

Remark 2 If we use the black circle connective $\bullet(\cdot)$ of Logic of Formal Inconsistency such that **CI** proves $\vdash \neg^\circ A \leftrightarrow \bullet A$ (see [9] p. 44) the negation-dual rules of the left negation rules of **CI** acquire the following forms:

$$\frac{A, \Omega \vdash \Pi, \bullet A}{\Omega \vdash \Pi, \bullet A, \neg A} \neg - R3 \quad \frac{\bullet A, \Omega \vdash \Pi}{\Omega \vdash \Pi, \neg \bullet A} \neg - R4$$

where the antisymmetric link is particularly clear. ■

Definition 8 *Let \mathbf{U} be a paraconsistent sequent system in the language of Logic of Formal Inconsistency, such that applying to \mathbf{U} the standard translation t with $t(\neg^\circ A) \equiv \neg(A \wedge \neg A)$ a formulation of a subsystem of classical logic **LK** is obtained, and such that \mathbf{U} has both left and right diagonal negation rules. Then the negation-dual system $\mathbf{nd} - \mathbf{U}$ of \mathbf{U} is [**structural and positive rules of LK**] plus [**negation-dual rules of the diagonal negation rules of \mathbf{U}**]. ■*

Thus, the non-diagonal negation rules do not contribute to the system $\mathbf{nd} - \mathbf{U}$. This reflects our assumption on the major relevance of diagonal negation rules as to the constructive behaviour of a system, but also takes into account the hypothesis that the standard translation of \mathbf{U} must remain a subsystem of **LK**, so that the possible non-diagonal negation rules cannot be too strange, i.e cannot produce exceptionally relevant information: e.g., they cannot produce the negation inconsistency of \mathbf{U} .

We can give the first formal condition for the definition of constructive paraconsistency:

Definition 9 *Let \mathbf{U} be a paraconsistent sequent system in the language of Logic of Formal Inconsistency, such that the negation-dual system $\mathbf{nd} - \mathbf{U}$ can be defined. Then we say that \mathbf{U} is pseudo-constructive if \mathbf{U} has cut-elimination and $\mathbf{nd} - \mathbf{U}$ is pseudo-intuitionistic. ■*

Then, in general, a bit of work is needed in order to establish the pseudo-constructivity of a system. It is easy to see that the above mentioned system **LDJ** presented in [25] is pseudo-constructive. But the interesting point is to investigate the constructivity of paraconsistent systems whose diagonal negation rules have the constraint formula property. As already remarked, the constraint formula declaratively expresses the distance between the sys-

tem \mathbf{U} and classical logic \mathbf{LK} , conveying effective information. Therefore, the constraint formula property must mark a difference in the hierarchy of constructive paraconsistency:

Definition 10 *Let \mathbf{U} be a pseudo-constructive paraconsistent system. Then, \mathbf{U} is declaratively constructive if it has left diagonal negation rules with the constraint formula property. ■*

We concentrate on \mathbf{CI} and we shall prove it is a declaratively constructive system.

Lemma 9 *The negation-dual system $\mathbf{nd} - \mathbf{CI}$ of \mathbf{CI} is given by:*

[**structural and positive rules of LK**] *plus*

$$\frac{A, \Omega \vdash \Pi, \neg^\circ A}{\Omega \vdash \Pi, \neg^\circ A, \neg A} \neg - R3 \quad \text{plus}$$

$$\frac{\neg^\circ A, \Omega \vdash \Pi}{\Omega \vdash \Pi, \neg \neg^\circ A} \neg - R4 \quad \text{plus}$$

$$\frac{X \vdash Y, B}{\neg B, X \vdash Y} \neg - L \quad \blacksquare$$

Proposition 9 *$\mathbf{nd} - \mathbf{CI}$ admits cut-elimination.*

Proof. The critical cases of cut occurrences to be considered are that where the cut formula is introduced by negation rules. These cases are analogous to those already examined for the cut-elimination in \mathbf{BC} and \mathbf{CI} , where constrained negation rules are involved. Then, we refer the reader to the proof of cut-elimination for \mathbf{BC} and \mathbf{CI} (Sections 3,4, Theorems 1,2). ■

Proposition 10 *$\mathbf{nd} - \mathbf{CI}$ cannot prove infinitely many instances of the excluded middle principle $B \vee \neg B$ and of the left double negation principle $\neg \neg B \rightarrow B$, so that it is a pseudo-intuitionistic system.*

Proof. By cut-elimination, it is easy to see that sequents $\vdash p \vee \neg p$ and $\neg \neg p \vdash p$, p propositional letter, cannot be $\mathbf{nd} - \mathbf{CI}$ provable. Analogously, it is straightforward to find arbitrarily complex B 's such that $\vdash B \vee \neg B$ and $\neg \neg B \vdash B$ cannot be provable without cuts. ■

Theorem 4 *\mathbf{CI} is a declaratively constructive paraconsistent system. ■*

In order to establish a complete notion of constructivity for paraconsistent logic, we deem that a further condition on the negation connective must be imposed. That is, we recall the wellknown idea, which is seminal in constructive mathematics, stating that *the negation of A must express an unprovability*

condition of A . In the next sections we present an *arithmetical semantics* for paraconsistent \mathbf{C} -systems that can help us to formalize such *negation as unprovability condition* (NUC).

Definition 11 *Let \mathbf{U} be a paraconsistent sequent system in the language of Logic of Formal Inconsistency. An arithmetical interpretation of the \mathbf{U} -language is a triple $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ where \mathbf{N} is the standard model of classical Arithmetic \mathbf{PA} , and φ is an application such that atomic propositional formulas p_i, p_r, \dots are sent into \mathbf{PA} -formulas of the forms $\text{Con}\mathbf{W}_j, \neg\text{Con}\mathbf{W}_k, \dots$, i.e. $\exists x \text{Pr ov}_{\mathbf{W}_j}(x, \#0 = 1), \neg\exists x \text{Pr ov}_{\mathbf{W}_k}(x, \#0 = 1), \mathbf{W}_j, \mathbf{W}_k$ consistent axiomatizable extensions of \mathbf{PA} , and for each compound formula B , $\varphi(B)$ is a formula of \mathbf{PA} -Provability Logic. B is true in $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ if $\mathbf{N} \models \varphi(B)$ in the standard sense. $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ is an arithmetical model of \mathbf{U} if $\mathbf{N} \models \varphi(A)$ for each \mathbf{U} -theorem A . ■*

Such definition will be discussed, specified and developed in Section 7. As we shall see in Section 8, \mathbf{BC} , \mathbf{CI} , \mathbf{CIL} admit arithmetical models. Arithmetical models provide a strong semantical tool in order to characterize the NUC-property:

Definition 12 *Let \mathbf{U} be a declaratively constructive paraconsistent system. Then \mathbf{U} is canonically constructive if it has the following formal NUC-property: \mathbf{U} admits an arithmetical model $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ such that for infinitely many non-atomic formulas B of \mathbf{U} we have $\varphi(\neg B) \equiv \neg \text{Pr}_{\mathbf{PA}}(\varphi(B))$. ■*

Proposition 11 *\mathbf{CI} is a canonically constructive paraconsistent system.*

Proof. See the construction of the arithmetical models of \mathbf{CI} in Section 8. ■

In a canonically constructive system, the formal NUC-property ensures that negation is completely controlled from a constructive standpoint: in particular, also that weak negation properties introduced by possible non-diagonal negation rules must fall under the action field of the formal NUC-property. It can be interesting to remark that the pseudo-constructive paraconsistent systems \mathbf{LDJ} [25] and \mathbf{DI} [1] are falsified by arithmetical models of \mathbf{BC} , \mathbf{CI} , \mathbf{CIL} . For example, $\neg((p \rightarrow p) \wedge \neg(p \rightarrow p))$ is an \mathbf{LDJ} -theorem which is false in such models. This could support the conjecture that the constraint formula property is a necessary condition for the formal NUC-property, but it is only a conjecture.

7 The meaning of the paraconsistent negation: arithmetical models

As already declared, we will focus on the *arithmetical semantics* of *propositional connectives* of **LFIs**. We recall the basic notions of Provability Logic that are employed in the sequel. We refer to [2,19,22] for a general presentation of such field, and to [15–18] for the proof theoretic approach. As well-known, for each recursively axiomatized system \mathbf{T} (and then for classical Arithmetic **PA** too) a binary primitive recursive predicate $Prov_{\mathbf{T}}(.,.)$ can be defined in Arithmetic **PA**, such that $Prov_{\mathbf{T}}(m,n)$ holds iff m is the Gödel-number of a \mathbf{T} -proof of the sentence with Gödel-number n . The **PA**-formula $\exists y Prov_{\mathbf{T}}(y, \#B)$ means “the sentence B is \mathbf{T} -provable” and we also write it as $Pr_{\mathbf{T}}(B)$. For brevity, we will omit the prefix $\#$ in the terms of the form $\#B$ (i.e. the Gödel-number of B) occurring as arguments of the provability predicate $Pr_{\mathbf{T}}(.)$. The following sequents, that express properties of the provability predicate, are **PA**-provable : (D1) $Pr_{\mathbf{T}}(A \rightarrow B) \vdash Pr_{\mathbf{T}}(A) \rightarrow Pr_{\mathbf{T}}(B)$; (D2) $Pr_{\mathbf{T}}(A \rightarrow B) \wedge Pr_{\mathbf{T}}(A) \vdash Pr_{\mathbf{T}}(B)$; (D3) $\vdash Pr_{\mathbf{T}}(A \wedge B) \leftrightarrow Pr_{\mathbf{T}}(A) \wedge Pr_{\mathbf{T}}(B)$; (D4) $Pr_{\mathbf{T}}(B) \vdash Pr_{\mathbf{T}}(Pr_{\mathbf{T}}(B))$. **PA** also proves the following useful relation (D5): $\vdash Pr_{\mathbf{T}}(\neg Pr_{\mathbf{T}}(B)) \leftrightarrow Pr_{\mathbf{T}}(0 = 1)$, which is a specific relation of classical **PA**-based Provability Logic, and that does not hold, in general, in Paraconsistent Provability Logic. As usual, in the classical setting, $\neg Pr_{\mathbf{T}}(0 = 1)$ and $Pr_{\mathbf{T}}(0 = 1)$ are canonically employed to state, respectively, the consistency and the inconsistency of \mathbf{T} , and are also written as $Con\mathbf{T}$, $\neg Con\mathbf{T}$, assuming that \mathbf{T} includes the arithmetical language. We also call $Con\mathbf{T}$, $\neg Con\mathbf{T}$ *global consistency assertions*, when we compare them with the *local consistency assertions* of the form ${}^\circ B$. The relationships between global and local consistency assertions are investigated by Paraconsistent Provability Logic based on **LFIs**’, which has been introduced in [4], [14].

The crucial points are the interpretations of the negation connective \neg and of the local consistency connective ${}^\circ(.)$ as *intensional connectives*. We have that, in general, $\varphi(\neg B) \equiv \neg Pr_{\mathbf{PA}}(\varphi(B))$: that is, the paraconsistent negation of B has the same meaning as $\langle\langle B \text{ is not provable in classical } \mathbf{PA} \rangle\rangle$ and it is a Π_1 -complexity operator. The interpretation of ${}^\circ(.)$ is linked to that of \neg , and, in general, $\varphi({}^\circ B) \equiv \neg Pr_{\mathbf{PA}}(\varphi(B) \wedge \neg Pr_{\mathbf{PA}}(\varphi(B)))$: that is, φ translates ${}^\circ(.)$ following the intended metatheoretical expression of ${}^\circ B$ as “ $\langle\langle B$ and not $B \rangle\rangle$ does not hold”, and moreover taking into account the mentioned interpretation of \neg . As natural, B is true in $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ if $\mathbf{N} \models \varphi(B)$ in the standard sense. However, we must underline that *arithmetical semantics*, even if compositional, is not uniform w.r.t. the syntactical structure, since it is a complexity-sensitive semantics. For example, in the *general arithmetical interpretations* (Definition 13) φ on circled propositional letters is $\varphi({}^\circ p_j) \equiv \neg Con\mathbf{PA}$, whereas $\varphi({}^\circ B) \equiv \neg Pr_{\mathbf{PA}}(\varphi(B) \wedge \neg Pr_{\mathbf{PA}}(\varphi(B)))$, which is **PA**-equivalent to $Con\mathbf{PA}$, if B is not a positive classical formula: that is, φ separates the meaning of ${}^\circ(.)$ applied on formulas that does not include the in-

tensional connectives $\neg, \circ(\cdot)$, from the meaning of $\circ(\cdot)$ applied on formulas that include intensional connectives, i.e. *the increasing of the wealth of information conveyed by the syntax changes the meaning attribution*.

Definition 13 Let $\mathbf{PA}^* \equiv \mathbf{PA} + \mathbf{ConPA}$, $\mathbf{W}_1 \equiv \mathbf{PA}^* + \mathbf{ConPA}^*$, $\mathbf{W}_2 \equiv \mathbf{W}_1 + \mathbf{ConW}_1, \dots$. Then a general arithmetical interpretation of the **C**-system language is a triple $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ where \mathbf{N} is the standard model of classical Arithmetic \mathbf{PA} , and φ is an injective application from the set $\{p_1, p_2, \dots\}$ of propositional letters into the set of sentences $\{\neg \mathbf{ConW}_j, \mathbf{ConW}_k : k, j = 1, 2, \dots\}$. φ is so specified on non-atomic formulas:

1. As to positive classical connectives ∇ , φ behaves standardly, that is $\varphi(A \nabla B) \equiv \varphi(A) \nabla \varphi(B)$ for any pair A, B of formulas.

2. As to the negation connective \neg and the local consistency connective $\circ(\cdot)$ we distinguish several cases:

2.1 Elementary cases. Let F be a non-atomic positive classical formula and $k \geq 0$:

$$\varphi(\neg p_j) \equiv \neg \text{Pr}_{\mathbf{PA}}(\varphi(p_j)) \quad \varphi(\neg F) \equiv \neg \text{Pr}_{\mathbf{PA}}(\varphi(F) \wedge \mathbf{ConPA})$$

$$\varphi(\circ p_j) \equiv \neg \mathbf{ConPA} \quad \varphi(\circ F) \equiv \neg \mathbf{ConPA}$$

$$\varphi(\neg^{2k+1}(\circ p_j)) \equiv \mathbf{ConPA} \quad \varphi(\neg^{2k+1}(\circ F)) \equiv \mathbf{ConPA}$$

$$\varphi(\neg^{2k}(\circ p_j)) \equiv \neg \mathbf{ConPA} \quad \varphi(\neg^{2k}(\circ F)) \equiv \neg \mathbf{ConPA}$$

2.2 Compound cases:

2.2.1 If B is not a positive classical formula and $k \geq 1$ then:

$$\varphi(\circ B) \equiv \neg \text{Pr}_{\mathbf{PA}}(\varphi(B) \wedge \neg \text{Pr}_{\mathbf{PA}}(\varphi(B)))$$

$$\text{for each } k \geq 0, \quad \varphi(\neg^{2k+1}(\circ B)) \equiv \text{Pr}_{\mathbf{PA}}(\neg \text{Pr}_{\mathbf{PA}}(\varphi(B) \wedge \neg \text{Pr}_{\mathbf{PA}}(\varphi(B))))$$

$$\text{for each } k \geq 1, \quad \varphi(\neg^{2k}(\circ B)) \equiv \neg \text{Pr}_{\mathbf{PA}}(\neg \text{Pr}_{\mathbf{PA}}(\varphi(B) \wedge \neg \text{Pr}_{\mathbf{PA}}(\varphi(B))))$$

2.2.2 If B is not a positive classical formula and B has not the forms $\circ D$, $\neg^{2k+1}(\circ D)$, $\neg^{2k}(\circ D)$ then:

$$\varphi(\neg B) \equiv \neg \text{Pr}_{\mathbf{PA}}(\varphi(B)) \text{ if } \mathbf{N} \not\models \varphi(B)$$

$$\text{Pr}_{\mathbf{PA}}(\varphi(B)) \text{ if } \mathbf{N} \models \varphi(B) \text{ and } \mathbf{PA} \not\vdash \varphi(B)$$

$$\neg \text{Pr}_{\mathbf{PA}}(\varphi(B)) \text{ if } \mathbf{PA} \vdash \varphi(B)$$

3. We say that a formula B of the **C**-system language is true in $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$,

and that $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ is an arithmetical model of B , if $\mathbf{N} \models \varphi(B)$. ■

Such definition formalizes the following seminal ideas:

1) The paraconsistent negation $\neg B$ has an intensional character and a constructive nature: this is expressed, *in general*, by an interpretation of the form $\neg \text{Pr}_{\mathbf{PA}}(\varphi(B))$. That is, the meaning is the negation of the provability of (the interpretation of) B in classical Arithmetic. Therefore, the paraconsistent negation is interpreted as a Π_1 operator. Of course, our general principle must nevertheless take into account the different cases arising by the consideration of the \mathbf{N} -truth and the \mathbf{PA} -provability of $\varphi(B)$: this is the reason of the three sub-cases that defines $\varphi(\neg B)$ for a non-elementary B .

2) The interpretation of the local consistency connective $\circ(\cdot)$ is such that, *in general*, $\varphi(\circ B)$ reflects both its metatheoretical intended meaning as “ $\ll B$ and not $B \gg$ does not hold” and the previously stated interpretation of the paraconsistent negation. Thus, in the most general case, the local consistency assertion $\circ B$ becomes a global consistency assertion on \mathbf{PA} , linked to the formula $\varphi(B)$: indeed, each true sentence of the form $\neg \text{Pr}_{\mathbf{PA}}(G)$ also asserts the consistency of \mathbf{PA} for any G .

3) The semantics must separate the role of formulas including the intensional connectives \neg and $\circ(\cdot)$ from that of formulas which do not include intensional connectives. This is the reason of the distinction between elementary and compound cases, and leads to remarkable differences between the corresponding interpretations. For example $\varphi(\circ B)$ is \mathbf{PA} -equivalent to the consistency ConPA of \mathbf{PA} for each non-elementary B , whereas $\varphi(\circ p_j)$ is $\neg \text{ConPA}$. The underlying idea is that formulas $\circ F$ with F without occurrences of \neg or $\circ(\cdot)$ cannot reliably express any true statement on the consistency of a system, even in the limit case where the system is reduced to a formula. Similarly, if F is positive classical, $\varphi(\neg F)$ is always simplified to a global consistency statement of the form $\neg \text{Pr}_{\mathbf{PA}}(\varphi(F) \dots)$: that is, it is assumed that the information included in F does not suffice to separate the interpretation of the negation of F into different meanings. Then, we can say that the *qualitative* complexity of the interpreted formula affects its meaning in the arithmetical semantics.

If E is any expression of the \mathbf{C} -system language we also write E^φ for $\varphi(E)$.

Proposition 12 *Let $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ be any general arithmetical interpretation. Suppose that B is not positive classical and $k \geq 0$. Then \mathbf{PA} proves:*

- i) $\varphi(\circ B) \longleftrightarrow \text{ConPA}$;
- ii) $\varphi(\neg^{2k+1}(\circ B)) \longleftrightarrow \neg \text{ConPA}$;
- iii) $\varphi(\neg^{2k}(\circ B)) \longleftrightarrow \text{ConPA}$.

Proof. i) $\varphi(\circ B) \equiv \neg \text{Pr}_{\mathbf{PA}}(\varphi(B) \wedge \neg \text{Pr}_{\mathbf{PA}}(\varphi(B)))$ is:

$$\neg(\text{Pr}_{\mathbf{PA}}(\varphi(B) \wedge \neg \text{Pr}_{\mathbf{PA}}(\varphi(B))))$$

which, by standard Provability Logic (D3), gives:

$$\neg(\text{Pr}_{\mathbf{PA}}(\varphi(B)) \wedge \text{Pr}_{\mathbf{PA}}(\neg \text{Pr}_{\mathbf{PA}}(\varphi(B)))) \quad \text{and then, by (D5),}$$

$$\neg(\text{Pr}_{\mathbf{PA}}(\varphi(B)) \wedge \text{Pr}_{\mathbf{PA}}(0 = 1)) \quad \text{from which, by (D3),}$$

$$\neg \text{Pr}_{\mathbf{PA}}(\varphi(B) \wedge 0 = 1) \quad \text{that is } \neg \text{Pr}_{\mathbf{PA}}(0 = 1), \text{ i.e. } \text{Con}\mathbf{PA}.$$

ii) By definition $\varphi(\neg^{2k+1}(\circ B))$ is:

$$\text{Pr}_{\mathbf{PA}}(\neg \text{Pr}_{\mathbf{PA}}(\varphi(B) \wedge \neg \text{Pr}_{\mathbf{PA}}(\varphi(B))))$$

that, by point i), is \mathbf{PA} -equivalent to $\text{Pr}_{\mathbf{PA}}(\neg \text{Pr}_{\mathbf{PA}}(0 = 1))$

which, by (D5), gives $\text{Pr}_{\mathbf{PA}}(0 = 1)$ that is $\neg \text{Con}\mathbf{PA}$.

iii) $\varphi(\neg^{2k}(\circ B))$ is by definition:

$$\neg \text{Pr}_{\mathbf{PA}}(\neg \text{Pr}_{\mathbf{PA}}(\varphi(B) \wedge \neg \text{Pr}_{\mathbf{PA}}(\varphi(B))))$$

that, analogously to points i) e ii), gives $\neg \text{Pr}_{\mathbf{PA}}(0 = 1)$. ■

It is worth noting that the above stated equivalence *does not hold* in general in Paraconsistent Provability Logic based on paraconsistent arithmetics \mathbf{PCA} or \mathbf{PACI} , that have been introduced in [4], [14]. That is, *from the standpoint of the object system*, which also could be paraconsistent Arithmetic \mathbf{PCA} or \mathbf{PACI} , interpreting $\circ B$ through $\neg \text{Pr}_{\mathbf{PA}}(\varphi(B) \wedge \neg \text{Pr}_{\mathbf{PA}}(\varphi(B)))$ is *not the same* as interpreting $\circ B$ through $\text{Con}\mathbf{PA}$. Such identification is only possible *a posteriori* in the classical setting.

We recall that $\text{Pr}_{\mathbf{PA}}(A) \rightarrow A$ is the local reflection principle for \mathbf{PA} (see [Smorynski] sec. 4.1). We write $\mathbf{Rfn}(\Sigma_1)$ for the restriction of the principle to Σ_1 -formulas, and note that $\text{Pr}_{\mathbf{PA}}(B)$ is a Σ_1 -formula.

Proposition 13 *Let $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ be any general arithmetical interpretation and suppose that B is not positive classical. Then:*

j) If $k \geq 0$, \mathbf{PA} proves $\varphi(\neg^{2k+1}(\circ B)) \longleftrightarrow \text{Pr}_{\mathbf{PA}}(\varphi(\neg^{2k}(\circ B)))$;

jj) If $k \geq 1$, the system $\mathbf{W} \equiv \mathbf{PA} + \mathbf{Rfn}(\Sigma_1)$ proves:

$$\varphi(\neg^{2k}(\circ B)) \longleftrightarrow \neg \text{Pr}_{\mathbf{PA}}(\varphi(\neg^{2k-1}(\circ B))).$$

Proof. j) By Proposition 12, $\varphi(\neg^{2k}(\circ B))$ is \mathbf{PA} -equivalent to $\text{Con}\mathbf{PA}$ i.e. to

$\neg \text{Pr}_{\text{PA}}(0 = 1)$ for each $k \geq 0$. Then,

$\text{Pr}_{\text{PA}}(\varphi(\neg^{2k}(\circ B)))$ is **PA**-equivalent to:

$$\text{Pr}_{\text{PA}}(\neg \text{Pr}_{\text{PA}}(0 = 1))$$

which, by (D5), is **PA**-equivalent to $\text{Pr}_{\text{PA}}(0 = 1)$ which is $\neg \text{ConPA}$. By Proposition 12: $\neg \text{ConPA} \longleftrightarrow \varphi(\neg^{2k+1}(\circ B))$ for each $k \geq 0$,

and we have the thesis.

jj) By Proposition 12, for each $k \geq 1$, $\varphi(\neg^{2k-1}(\circ B))$ is **PA**-equivalent to $\neg \text{ConPA}$, so that

$\neg \text{Pr}_{\text{PA}}(\varphi(\neg^{2k-1}(\circ B)))$ is **PA**-equivalent to:

$$\neg \text{Pr}_{\text{PA}}(\text{Pr}_{\text{PA}}(0 = 1)).$$

By (D4), which is $\text{Pr}_{\text{PA}}(B) \rightarrow \text{Pr}_{\text{PA}}(\text{Pr}_{\text{PA}}(B))$, we have:

$$\neg \text{Pr}_{\text{PA}}(\text{Pr}_{\text{PA}}(0 = 1)) \rightarrow \neg \text{Pr}_{\text{PA}}(0 = 1).$$

On the other hand, by **Rfn**(Σ_1), which gives:

$\text{Pr}_{\text{PA}}(\text{Pr}_{\text{PA}}(0 = 1)) \rightarrow \text{Pr}_{\text{PA}}(0 = 1)$, we have:

$$\neg \text{Pr}_{\text{PA}}(0 = 1) \rightarrow \neg \text{Pr}_{\text{PA}}(\text{Pr}_{\text{PA}}(0 = 1)).$$

Thus, $\neg \text{Pr}_{\text{PA}}(\text{Pr}_{\text{PA}}(0 = 1))$ is **W**-equivalent to $\neg \text{Pr}_{\text{PA}}(0 = 1)$, which is **ConPA**.

Then, by Proposition 12, it is **W**-equivalent to $\varphi(\neg^{2k}(\circ B))$.

Observe that the implication:

$$\neg \text{Pr}_{\text{PA}}(\varphi(\neg^{2k-1}(\circ B))) \rightarrow \varphi(\neg^{2k}(\circ B))$$

is **PA**-provable. ■

We note that, by Proposition 13, for any B which is not positive classical, $\varphi(\neg^{2k}(\circ B))$ is **N**-true and not **PA**-provable, and $\varphi(\neg^{2k+1}(\circ B))$ is **N**-false. If we moreover consider the equivalences stated by Proposition 12, we may assume that the cases 2.2.1 and 2.2.2 of Definition 13 can be unified, from the standpoint of classical Arithmetic providing the models, without losing the established essential meaning of $\varphi(\circ B)$ and $\varphi(\neg B)$:

Definition 14 *A simplified general arithmetical interpretation of the **C**-system language is defined as in Definition 13, with the replacement of the clause 2.2 by the following:*

Compound cases:

If B is not a positive classical formula then:

$$\varphi(\circ B) \equiv \text{ConPA}$$

If B is not atomic and does not belong to the elementary cases (clause 2.1, Definition 1), then:

$$\begin{aligned} \varphi(\neg B) \equiv & \quad \neg \text{Pr}_{\text{PA}}(\varphi(B)) \text{ if } \mathbf{N} \not\models \varphi(B) \\ & \text{Pr}_{\text{PA}}(\varphi(B)) \text{ if } \mathbf{N} \models \varphi(B) \text{ and } \mathbf{PA} \not\vdash \varphi(B) \\ & \neg \text{Pr}_{\text{PA}}(\varphi(B)) \text{ if } \mathbf{PA} \vdash \varphi(B) \end{aligned} \quad \blacksquare$$

In the sequel we briefly say *general arithmetical interpretation* for any interpretation φ defined in Definition 14, since we will always work with *simplified* interpretations.

In the next section we shall see that arithmetical models really are models for **C**-systems.

8 Arithmetical models for **BC**, **CI**, **CIL** and the semantical complexity gap between the paraconsistent negation \neg and the local consistency connective $\circ(\cdot)$

In this Section it will be proven that each system among **BC**, **CI**, **CIL** has a denumerable infinity of arithmetical models. However, while all *general* arithmetical interpretations (Definition 13) provide arithmetical models for the system given by the positive part of **PC** *plus* the rule set

$\{\neg - L1, \neg - L3, \neg - L4, \neg - L5, \neg - R\}$, that is for the system only expressing the paraconsistent negation from **BC** to **CIL**, we have a partial break when the local consistency connective $\circ(\cdot)$ is considered: indeed, general arithmetical interpretations can falsify the $\circ(\cdot)$ -introduction rule *RCi* (Proposition 15).

For example, consider any *RCi*-instance $\frac{B \wedge \neg B, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \circ B}$ with B positive classical formula, and take the simplest case:

$$\frac{\frac{p_j \vdash p_j}{p_j \wedge \neg p_j \vdash p_j}}{\vdash p_j, \circ p_j}$$

where B is the letter p_j . Then, if we choose $\varphi(p_j) \equiv \neg \text{ConW}_j$, by definition, we have $\varphi(\circ B) \equiv \neg \text{ConPA}$ for each φ , which is false, so that $(\vdash p_j, \circ p_j)^\varphi$ is

false in \mathbf{N} . Through analogous considerations, we have the same conclusion for the RCi -instances where B in ${}^\circ B$ is an arbitrarily complex positive classical formula. To model the RCi -rule too, we must restrict the set of general arithmetical interpretations to the subset of *positive arithmetical interpretations*, such that $\varphi(p_j)$ has the form $Con\mathbf{W}_{kj}$ for each j , as Theorems 7 and 8 show. Thus, the set of interpretations which model the paraconsistent negation is larger than that modeling the local consistency connective. That is, in the arithmetical semantics setting, in general, the constraints on the truth of a \mathbf{CI} -theorem including ${}^\circ(\cdot)$ are stronger than the truth constraints of a \mathbf{CI} -theorem possibly including negation connectives but where ${}^\circ(\cdot)$ does not occur. We deem that this condition allows to state that the semantical complexity of ${}^\circ(\cdot)$ is higher than the semantical complexity of \neg . As a corroboration, we note that, even if both connectives are intensional, the intensionality of \neg can be seen as simpler: in some cases the paraconsistent negation has a completely classical behaviour, as the rule $\neg - R$ shows. Differently, ${}^\circ(\cdot)$ is a purely intensional connective with a *metalogical content*, since it expresses a *consistency* condition.

Furthermore, a development can be added to this picture: if in the RCi -principal formula ${}^\circ B$ the subformula B is not positive classical, then the RCi -instance is verified under *all* the arithmetical interpretations. Indeed the system $\mathbf{CIL}^* \equiv \mathbf{BC} + \neg - L4 + \neg - L5 + RCi^*$, where RCi^* is the restriction of RCi mentioned above, has a model $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ for *each general* arithmetical interpretation φ (Theorem 6). A heuristic explanation could be the following. We have already observed that arithmetical semantics is an information-sensitive semantics, i.e. it takes into account the quality of the information included in the interpreted formulas; therefore, in some sense, it selects as problematical those rules introducing formulas ${}^\circ B$ where no negation connectives occur in B , in this way emphasizing the evident but deep fact that consistency statements, even if minimal and local, cannot be established without using negation.

We recall that a \mathbf{PA} -sequent $W \vdash Z$ is \mathbf{N} -true if and only if either at least one formula of Z is \mathbf{N} -true or at least one formula of W is \mathbf{N} -false. If P is a proof-tree in the \mathbf{C} -system language we write P^φ for the tree obtained by replacing each sequent $X \vdash Y$ of P by $(X \vdash Y)^\varphi$. Given any sequent $\Gamma \vdash \Delta$, we write $\wedge\Gamma$ for the conjunction of the formulas of Γ , and $\vee\Delta$ for the disjunction of the formulas of Δ .

Proposition 14 *Let \mathcal{G} be any classical logical rule introducing a positive connective. Then \mathcal{G} is sound w.r.t. each general arithmetical interpretation $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$.*

Proof. It is straightforward to see that if the \mathcal{G} -premises are true, the conclusion must be true. For example, consider a $\vee - R$ rule occurrence in a \mathbf{BC} -proof

$P: \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, B \vee A}$. Assuming that $\Gamma^\varphi \vdash \Delta^\varphi, A^\varphi$ is \mathbf{N} -true, $\Gamma^\varphi \vdash \Delta^\varphi, B^\varphi \vee A^\varphi$ must be \mathbf{N} -true. ■

Definition 15 Let \mathcal{R} be any logical rule of a system $\mathbf{V} \in \{\mathbf{BC}, \mathbf{CI}, \mathbf{CIL}, \mathbf{LK}, \mathbf{LJ}\}$. Then, a proof with a minimal length for \mathcal{R} in \mathbf{V} is any cut free \mathbf{V} -proof Q such that: Q has \mathcal{R} as end-rule, each auxiliary formula of the end-rule is the successor of an axiom formula, and no shorter proof M exists whose end-sequent is a sub-sequent of the Q -root. ⁴ ■

Lemma 10 Let \mathcal{R} be an element of the set $\{\neg - L1, \neg - L3, \neg - L4, \neg - L5, \neg - R\}$ and let Q be any proof with a minimal length (Definition 15) for \mathcal{R} in the system $\mathbf{BC} + \neg - L4 + \neg - L5$. Then the end-sequent of Q^φ is true for each general arithmetical interpretation $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$.

Proof. For each rule we proceed by induction on the complexity of the auxiliary formula, as it is described by Definition 14.

1. $\neg - L1$: the proofs Q of minimal length have the form: $\frac{A \vdash A}{\neg \neg A \vdash A}$.

1.1. Atomic and elementary cases of the auxiliary formula A :

1.1.1 Let $A \in \{p, F\}$, p propositional letter and F any non-atomic positive classical formula. Observe that $\neg \neg F$ is not elementary in the taxonomy established by Definitions 13 and 14. We note that $\varphi(\neg A) \equiv \neg \text{Pr}_{\mathbf{PA}}(E)$ with E not \mathbf{PA} -provable, which is \mathbf{N} -true and not \mathbf{PA} -provable for each φ . Then $\varphi(\neg \neg A) \equiv \text{Pr}_{\mathbf{PA}}(\neg \text{Pr}_{\mathbf{PA}}(\varphi(A)))$ which is \mathbf{N} -false, and the conclusion of Q^φ is true for each φ .

1.1.2 Let $A \in \{\neg p, \neg F\}$, p and F as above. $\varphi(A) \equiv \neg \text{Pr}_{\mathbf{PA}}(E)$ with E not \mathbf{PA} -provable, which is \mathbf{N} -true and not \mathbf{PA} -provable for each φ . Then $(\neg \neg A \vdash A)^\varphi$ is true for each φ .

1.1.3 Let $A \in \{\circ p, \circ F\}$, p and F as above. By Definitions 13 and 14, $\varphi(\neg \neg A) \equiv \neg \text{Con}_{\mathbf{PA}}$ for each φ , which is \mathbf{N} -false. Then the conclusion is true for each φ .

1.1.4 Let $A \in \{\neg^h(\circ p), \neg^h(\circ F)\}$, p and F as above, $h \geq 1$. By construction $\varphi(\neg^{h+2}(\circ p)) \equiv \varphi(\neg^h(\circ p)), \varphi(\neg^{h+2}(\circ F)) \equiv \varphi(\neg^h(\circ F))$ and the conclusion is true for each φ .

1.2 Compound cases of the auxiliary formula A :

⁴ Observe that, by alternating redundant weakenings and contractions, one can produce arbitrarily long cut free proofs with the same axioms and the same conclusion.

suppose that $\varphi(A)$ is \mathbf{N} - true; then the conclusion is true. Suppose that $\varphi(A)$ is \mathbf{N} - false; then, by definition of φ on compound cases, $\varphi(\neg A) \equiv \neg \text{Pr}_{\mathbf{PA}}(\varphi(A))$ which is true and not \mathbf{PA} -provable, and then $\varphi(\neg\neg A) \equiv \text{Pr}_{\mathbf{PA}}(\varphi(\neg A))$ which is false, and the conclusion is true. Thus, the conclusion of Q^φ is true for each φ .

2. $\neg - L3$: the proofs Q of minimal length have the form:
$$\frac{A \vdash A}{\frac{\circ A, A \vdash A}{\circ A, \neg A, A \vdash}} .$$
 We preliminarily establish that the $\neg - L3$ instances with auxiliary formula of the form $\circ B$ are considered as $\neg - L4$ instances (see point 3.).

2.1 Atomic and elementary cases of the auxiliary formula A :

2.1.1 Let $A \in \{p, F\}$, p propositional letter and F any non-atomic positive classical formula. Then $\varphi(\circ A) \equiv \neg \text{Con}_{\mathbf{PA}}$ for each φ , which is \mathbf{N} - false. Then $(\circ A, \neg A, A \vdash)^\varphi$ is true for each φ .

2.1.2 Let $A \in \{\neg p, \neg F\}$, p and F as above. We have $\varphi(A) \equiv \neg \text{Pr}_{\mathbf{PA}}(E)$ with E not \mathbf{PA} -provable, so that $\varphi(A)$ is true and not \mathbf{PA} -provable. Then $\varphi(\neg A) \equiv \text{Pr}_{\mathbf{PA}}(\varphi(A))$ which is false and the conclusion is true for each φ .

2.1.3 Let $A \in \{\neg^{2k+1}(\circ p), \neg^{2k}(\circ p), \neg^{2k+1}(\circ F), \neg^{2k}(\circ F)\}$, p and F as above, $k \geq 0$. By construction, in the antecedent of $(\circ A, \neg A, A \vdash)^\varphi$ a formula of the form $\neg \text{Con}_{\mathbf{PA}}$ always occurs, so that the conclusion is true for each φ .

2.2 Compound cases of the auxiliary formula A :

suppose that $\varphi(A)$ is false; then $(\circ A, \neg A, A \vdash)^\varphi$ is true. Suppose that $\varphi(A)$ is true and not \mathbf{PA} -provable; then $\varphi(\neg A) \equiv \text{Pr}_{\mathbf{PA}}(\varphi(A))$ which is false, and the conclusion is true. Suppose that $\varphi(A)$ is true and \mathbf{PA} -provable; then $\varphi(\neg A) \equiv \neg \text{Pr}_{\mathbf{PA}}(\varphi(A))$ which is false, and the conclusion is true. Thus, the conclusion of Q^φ is true for each φ .

3. $\neg - L4$: the proofs Q of minimal length have the form:
$$\frac{\circ A \vdash \circ A}{\neg \circ A, \circ A \vdash} .$$

3.1 Atomic and elementary cases of the auxiliary formula $\circ A$:

3.1.1 Let $\circ A \in \{\circ p, \circ F\}$, p propositional letter and F any non-atomic positive classical formula. Then $\varphi(\circ A) \equiv \neg \text{Con}_{\mathbf{PA}}$ for each φ , which is \mathbf{N} - false. Then $(\neg \circ A, \circ A \vdash)^\varphi$ is true for each φ .

3.2 Compound cases of the auxiliary formula $\circ A$:

by Definition 14, $\varphi(\circ A) \equiv \text{Con}_{\mathbf{PA}}$ which is true and not \mathbf{PA} -provable, so that $\varphi(\neg \circ A) \equiv \text{Pr}_{\mathbf{PA}}(\varphi(\circ A))$ which is false, and the conclusion is true for each φ .

4. $\neg - L5$: the proofs Q of minimal length have the following possible forms:

$$\frac{\frac{A \vdash A \quad \neg A \vdash \neg A}{A, \neg A \vdash A \wedge \neg A}}{\neg(A \wedge \neg A), A, \neg A \vdash} \qquad \frac{A \wedge \neg A \vdash A \wedge \neg A}{\neg(A \wedge \neg A), A \wedge \neg A \vdash}$$

since, for each φ , $(\neg(A \wedge \neg A), A, \neg A \vdash)^\varphi$ is true if and only if $(\neg(A \wedge \neg A), A \wedge \neg A \vdash)^\varphi$ is true, we examine the first case only.

4.1 Atomic and elementary cases of the auxiliary formula A :

4.1.1 Let $A \in \{p, F\}$, p propositional letter and F any non-atomic positive classical formula. Suppose that $\varphi(A)$ is false; then $(\neg(A \wedge \neg A), \neg A, A \vdash)^\varphi$ is true. Suppose that $\varphi(A)$ is true; since by definition $\varphi(\neg A)$ is true and not \mathbf{PA} -provable, $\varphi(A \wedge \neg A) \equiv \varphi(A) \wedge \varphi(\neg A)$ is true too and not \mathbf{PA} -provable; thus $\varphi(\neg(A \wedge \neg A)) \equiv \text{Pr}_{\mathbf{PA}}(\varphi(A \wedge \neg A))$ which is false, and the conclusion is true. Then, the conclusion of Q^φ is true for each φ .

4.1.2 Let $A \in \{\neg p, \neg F\}$, p and F as above. We conclude as in 2.1.2.

4.1.3 Let $A \in \{\neg^{2k+1}(\circ p), \neg^{2k}(\circ p), \neg^{2k+1}(\circ F), \neg^{2k}(\circ F)\}$, p and F as above, $k \geq 0$. We conclude as in 2.1.3.

4.2 Compound cases of the auxiliary formula A : we conclude as in 2.2.

5. $\neg - R$: the proofs Q of minimal length have the form: $\frac{A \vdash A}{\vdash A, \neg A}$.

5.1. Atomic and elementary cases of the auxiliary formula A :

5.1.1 Let $A \in \{p, F, \neg p, \neg F, \circ p, \circ F, \neg^{2k+1}(\circ p), \neg^{2k}(\circ p), \neg^{2k+1}(\circ F), \neg^{2k}(\circ F)\}$, p propositional letter and F any non-atomic positive classical formula. In these cases either $\varphi(A)$ or $\varphi(\neg A)$ is true for each φ and the conclusion is true for each φ .

5.2 Compound cases of the auxiliary formula A :

If $\varphi(A)$ is true the conclusion is true. If $\varphi(A)$ is false then $\varphi(\neg A) \equiv \neg \text{Pr}_{\mathbf{PA}}(\varphi(A))$ which is true, and the conclusion is true. Thus, the conclusion of Q^φ is true for each φ . ■

In the proof of the next Theorem 5, we often employ the fact that if F is a positive classical formula then a sub-class Φ of the class of general interpretations exists, such that F is true for each $\varphi \in \Phi$. Such point will be clarified by Lemma 11.

Theorem 5 *Let $X \vdash Y$ be the root of a cut-free proof P in the system $\mathbf{BC}+$*

$\neg - L4 + \neg - L5$ and $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ any general arithmetical interpretation. Then $(X \vdash Y)^\varphi$ is true.

Proof. We proceed by a main induction on the length of P , with a subinduction on the complexity of the auxiliary formulas of the rule occurrences in P , following the taxonomy given in Definition 14. The relevant rules are $\neg - L1, \neg - L3, \neg - L4, \neg - L5, \neg - R$; the remaining cases are straightforward, see Proposition 14.

Basis of the main induction: it is given by Lemma 10.

Step of the main induction: for each relevant rule occurrence \mathcal{R} in P , which is not the end-rule of a P sub-proof of minimal length for \mathcal{R} , we assume that the premise is true for each φ and prove that the conclusion is true for each φ . In general, the premise of \mathcal{R} has the form

$\{\text{possible constraint or auxiliary formulas}\}, \Gamma \vdash$

$\Delta, \{\text{possible auxiliary formula}\}$

and we will examine only the *non-trivial cases* for the thesis, that is: ($\wedge \Gamma^\varphi$ true and $\vee \Delta^\varphi$ false) or (Γ^φ empty and $\vee \Delta^\varphi$ false) or ($\wedge \Gamma^\varphi$ true and Δ^φ empty) or (Γ^φ empty and Δ^φ empty); in the remaining cases the truth of the conclusion does not depend on \mathcal{R} .

1. Let \mathcal{R} be a $\neg - L1$ occurrence in P : $\frac{B, \Gamma \vdash \Delta}{\neg \neg B, \Gamma \vdash \Delta}$. The induction hypothesis states that $B^\varphi, \Gamma^\varphi \vdash \Delta^\varphi$ is $\mathbf{N}-$ true for each φ . Then, in the non-trivial cases, B^φ is false for each φ .

1.1 Atomic and elementary cases of the auxiliary formula B of \mathcal{R} : if B^φ is false for each φ , then $B \in \{\circ p, \circ F, \neg^{2k}(\circ p), \neg^{2k}(\circ F)\}$, p propositional letter and F any non-atomic positive classical formula, $k \geq 1$. Thus $\neg \neg B \in \{\neg^{2h+2}(\circ p), \neg^{2h+2}(\circ F)\}$, $h \geq 0$, and $\varphi(\neg \neg B) \equiv \neg \text{ConPA}$ for each φ , which is false and the thesis holds.

1.2 Compound cases of the auxiliary formula B of \mathcal{R} : if B^φ is false for each φ , then $\varphi(\neg B) \equiv \neg \text{PrPA}(\varphi(B))$ which is true and not \mathbf{PA} -provable, and then $\varphi(\neg \neg B) \equiv \text{PrPA}(\varphi(\neg B))$ which is false, and the conclusion is true for each φ .

2. Let \mathcal{R} be a $\neg - L3$ occurrence in P : $\frac{\circ B, \Gamma \vdash \Delta, B}{\circ B, \neg B, \Gamma \vdash \Delta}$. Recall that the $\neg - L3$ instances with auxiliary formula of the form $\circ G$ are considered as $\neg - L4$ instances. The induction hypothesis states that $(\circ B, \Gamma \vdash \Delta, B)^\varphi$ is $\mathbf{N}-$ true for each φ .

2.1 Atomic and elementary cases of the auxiliary formula B of \mathcal{R} : B is compatible with the induction hypothesis if $B \in \{p, F, \neg p, \neg F, \neg^{2k+1}(\circ p), \neg^{2k+1}(\circ F)\}$, p propositional letter and F any non-atomic positive classical formula, $k \geq 0$. In these cases for each φ either $\varphi(\circ B)$ is false or $\varphi(B)$ is true. If $\varphi(\circ B)$ is false, the conclusion too is true. The cases where, for each φ , $\varphi(\circ B)$ is true and $\varphi(B)$ is true, are given by $B \in \{\neg p, \neg F, \neg^{2k+1}(\circ p), \neg^{2k+1}(\circ F)\}$. It easy to see, either by definition or by the examination of the formulas $\varphi(\neg\neg A)$ already produced at point 1 of Lemma 10, that $\varphi(\neg B)$ results as false for each φ , and the thesis holds.

2.2 Compound cases of the auxiliary formula B of \mathcal{R} : $\varphi(\circ B) \equiv \text{ConPA}$ for each φ , then the induction hypothesis forces, in the non-trivial cases, B^φ true for each φ . Then we conclude as in point 2.2 of the proof of Lemma 10.

3. Let \mathcal{R} be a $\neg - L4$ occurrence in P : $\frac{\Gamma \vdash \Delta, \circ B}{\neg \circ B, \Gamma \vdash \Delta}$. The induction hypothesis states that $(\Gamma \vdash \Delta, \circ B)^\varphi$ is $\mathbf{N}-$ true for each φ . Then, in the non-trivial cases, $\varphi(\circ B)$ must be true for each φ .

3.1 Atomic and elementary cases of the auxiliary formula $\circ B$ of \mathcal{R} : in such cases $\varphi(\circ B) \equiv \neg \text{ConPA}$ for each φ , which is false, so that they are not compatible with the induction hypothesis.

3.2 Compound cases of the auxiliary formula $\circ B$ of \mathcal{R} : in such cases $\varphi(\circ B) \equiv \text{ConPA}$ for each φ , which is true and not \mathbf{PA} -provable, so that $\varphi(\neg \circ B) \equiv \text{Pr}_{\mathbf{PA}}(\varphi(\circ B))$ which is false, and the conclusion is true for each φ .

4. Let \mathcal{R} be a $\neg - L5$ occurrence in P : $\frac{\Gamma \vdash \Delta, B \wedge \neg B}{\neg(B \wedge \neg B), \Gamma \vdash \Delta}$. The induction hypothesis states that $(\Gamma \vdash \Delta, B \wedge \neg B)^\varphi$ is $\mathbf{N}-$ true for each φ , and in the non-trivial cases this is the same as requiring that $\varphi(B \wedge \neg B)$ is true for each φ , that is $\varphi(B)$ true for each φ and $\varphi(\neg B)$ true for each φ . Thus: if $\varphi(B \wedge \neg B)$ is true and \mathbf{PA} -provable, then $\varphi(\neg(B \wedge \neg B)) \equiv \neg \text{Pr}_{\mathbf{PA}}(\varphi(B \wedge \neg B))$ which is false, and the conclusion is true; if $\varphi(B \wedge \neg B)$ is true and not \mathbf{PA} -provable, then $\varphi(\neg(B \wedge \neg B)) \equiv \text{Pr}_{\mathbf{PA}}(\varphi(B \wedge \neg B))$ which is false, and the conclusion is true. Then, the conclusion is true for each φ .

5. Let \mathcal{R} be a $\neg - R$ occurrence in P : $\frac{B, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg B}$. The induction hypothesis states that $(B, \Gamma \vdash \Delta)^\varphi$ is $\mathbf{N}-$ true for each φ . Then, in the non-trivial cases, $\varphi(B)$ must be false for each φ .

5.1 Atomic and elementary cases of the auxiliary formula B of \mathcal{R} : if B^φ is false for each φ , then $B \in \{\circ p, \circ F, \neg^{2k}(\circ p), \neg^{2k}(\circ F)\}$, p propositional letter and F any positive classical formula, $k \geq 1$. By definition, $\varphi(\neg B)$ is true for each φ , and the conclusion is true for each φ .

5.2 Compound cases of the auxiliary formula B of \mathcal{R} : if B^φ is false for each φ , then $\varphi(\neg B) \equiv \neg \text{Pr}_{\mathbf{PA}}(\varphi(B))$ is true and the conclusion is true for each φ . ■

Corollary 2 *Each general arithmetical interpretation $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ is a model of \mathbf{BC} . In particular, ${}^\circ A \rightarrow \neg(A \wedge \neg A)$ is true in each $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$. ■*

Corollary 3 *Each instance of the excluded middle principle $A \vee \neg A$, i.e. each sequent of the form $\vdash A, \neg A$, is true in each general arithmetical interpretation $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$.*

Proof. See point 5 of the proof of Lemma 10. ■

Proposition 15 *Let P be any proof of minimal length with an \mathbf{RCi} -instance as end-rule, such that the auxiliary formula $B \wedge \neg B$ is not introduced by axiom and the subformula B is positive classical. Then the end-sequent of P can be falsified by suitable general arithmetical interpretations.*

Proof. Consider the following \mathbf{CI} -proof P : $\frac{B \vdash B}{\frac{B \wedge \neg B \vdash B}{\vdash B, {}^\circ B}}$ where B is any positive classical formula such that $\varphi(B)$ is false for some general arithmetical interpretation $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$. There are infinitely many B with such property: for example take as B any letter p_j and choose $\varphi(p_j) \equiv \neg \text{Con}\mathbf{W}_j$. By definition, we have $\varphi({}^\circ B) \equiv \neg \text{Con}\mathbf{PA}$ for each φ , which is false, so that $(\vdash B, {}^\circ B)^\varphi$ is false. ■

Corollary 4 *The proper \mathbf{Ci} -axiom $\neg{}^\circ A \rightarrow A \wedge \neg A$ is falsified by suitable general arithmetical interpretations.*

Proof. Take the letter p for A and use the φ mentioned in the proof of Proposition 15. ■

Corollary 5 *The proper \mathbf{Cil} -axiom $\neg(A \wedge \neg A) \rightarrow {}^\circ A$ is falsified by suitable general arithmetical interpretations.*

Proof. Take the letter p for A and use the φ mentioned in the proof of Proposition 15. ■

Theorem 6 *Let $X \vdash Y$ be the root of a cut-free proof P in the system $\mathbf{CIL}^* \equiv \mathbf{BC} + \neg - L4 + \neg - L5 + \mathbf{RCi}^*$, where \mathbf{RCi}^* is the restriction of \mathbf{RCi} to auxiliary formulae $B \wedge \neg B$ such that B is not positive classical. Then $(X \vdash Y)^\varphi$ is true for each general arithmetical interpretation $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$.*

Proof. We add to the proof of Theorem 5 the following remark. Consider any \mathbf{RCi} -instance in a proof P :

$\frac{A \wedge \neg A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \circ A}$. If A is not positive classical, then $\varphi(\circ A) \equiv \text{ConPA}$ for each φ and $(\Gamma \vdash \Delta, \circ A)^\varphi$ is true for each φ . ■

Corollary 6 *Each general arithmetical interpretation $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ is a model of \mathbf{CIL}^* . ■*

Definition 16 *A positive arithmetical interpretation of the \mathbf{C} -system language is a general arithmetical interpretation $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ such that $\varphi(p_j)$ has the form ConW_{kj} for each j . ■*

Lemma 11 *Let $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ be a positive arithmetical interpretation and F a positive classical formula. Then $\varphi(F)$ is true.*

Proof. The thesis is obvious, with the following remark: the a priori assumption of the consistency of each \mathbf{W}_k , imposes the truth in \mathbf{N} of any sentence of the form $\text{ConW}_i \rightarrow \text{ConW}_h$, by properties of classical implication in tarskian semantics, even if \mathbf{W}_i is a proper subsystem of \mathbf{W}_h . ■

Theorem 7 *Let $X \vdash Y$ be the root of a \mathbf{CI} - proof P , and let $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ be any positive arithmetical interpretation. Then $(X \vdash Y)^\varphi$ is true.*

Proof. We proceed by a main induction on the length of P , with a subinduction on the complexity of the auxiliary formulas of the rule occurrences in P , following the taxonomy given in Definition 14.

Basis of the main induction: we add to the proof of the induction basis for Theorem 5 the case of the rule \mathbf{RCi} . The proofs Q with a minimal length for \mathbf{RCi} (Definition 15) may have the following forms:

$$\frac{A \vdash A}{A \wedge \neg A \vdash A} \quad \frac{\neg A \vdash \neg A}{A \wedge \neg A \vdash \neg A} \quad \frac{A \wedge \neg A \vdash A \wedge \neg A}{\vdash A \wedge \neg A, \circ A}$$

If A is not positive classical then $\varphi(\circ A) \equiv \text{ConPA}$ for each φ and the conclusions are true for each φ .

Let $A \in \{p, F\}$, p propositional letter, F any non atomic positive classical formula. In these cases both $\varphi(A)$ and $\varphi(\neg A)$ are true for each positive φ , and the conclusions are true for each positive φ .

Step of the main induction: for each rule occurrence \mathcal{R} in P , which is not the end-rule of a P sub-proof of minimal length for \mathcal{R} , we assume that the premise is true for each positive φ and prove that the conclusion is true for each positive φ . We only have to add to the proof of the induction step for Theorem 5 the \mathbf{RCi} case. In fact, it is possible to verify that each item of such proof can be performed analogously, by restricting the induction hypothesis to

the truth of the premise for each *positive* φ , and the thesis to the truth of the conclusion for each *positive* φ . Therefore, let \mathcal{G} be any *RCi*-rule occurrence in P : $\frac{B \wedge \neg B, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \circ B}$. We have to consider only the case with B positive classical. The induction hypothesis states that $(B \wedge \neg B, \Gamma \vdash \Delta)^\varphi$ is \mathbf{N} -true for each positive φ . Then, in the non-trivial cases, $\varphi(B \wedge \neg B)$ must be false for each positive φ . But if B is positive classical and φ positive, $\varphi(B \wedge \neg B)$ is true, and these cases are not compatible with the induction hypothesis. ■

Corollary 7 *Each positive arithmetical interpretation is a model of \mathbf{CI} .* ■

Theorem 8 *Let $X \vdash Y$ be the root of a \mathbf{CIL} -proof P , and let $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ be any positive arithmetical interpretation. Then $(X \vdash Y)^\varphi$ is true.*

Proof. The proof is obtained by assembling the proofs of Theorems 5 and 7. ■

Corollary 8 *Each positive arithmetical interpretation is a model of \mathbf{CIL} .* ■

Corollary 9 *Let $\mathbf{U} \in \{\mathbf{BC}, \mathbf{CIL}^*, \mathbf{CI}, \mathbf{CIL}\}$ and let $\mathcal{M} \equiv \langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ be any arithmetical model of \mathbf{U} . Then each bottom particle D of \mathbf{U} is false in \mathcal{M} .*

Proof. A \mathbf{U} -bottom particle (Section 2) is a formula D such that the sequent $D \vdash$ is \mathbf{U} -provable. By hypothesis $(D \vdash)^\varphi$ is true in \mathcal{M} and then $\varphi(D)$ must be false. ■

A proper completeness result w.r.t arithmetical models for a suitable \mathbf{CI} -extension is in progress. To this end, the presented interpretations must be further refined. For example, we will introduce models $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ such that $\varphi(\circ G)$ is false for some classes of non-elementary complex formulas G .

9 Metalogical completeness: falsifying classical logic and intuitionistic logic

We have a kind of *metalogical* completeness property of an important subclass of paraconsistent Logics of Formal Inconsistency w.r.t arithmetical models $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$: *indeed, arithmetical models verify \mathbf{BC} , \mathbf{CI} , \mathbf{CIL} but falsify classical logic \mathbf{LK} and intuitionistic logic \mathbf{LJ} .* If we observe that arithmetical models seem a constructive environment in the standard sense, since they essentially are the *provability logic* of \mathbf{PA} ⁵, this is a remarkably paradoxical

⁵ A notable simple fact is that classical Provability Logic substantially rejects the *excluded middle principle*, in the following sense: $Pr_{\mathbf{T}}(A) \vee Pr_{\mathbf{T}}(\neg A)$ does not hold in general, even if $Pr_{\mathbf{T}}(A \vee \neg A)$ trivially holds.

result, from which important considerations could arise on the role of Logics of Formal Inconsistency in constructive mathematics.

Proposition 16 *Let F be any arbitrarily complex positive classical formula, and let $\mathcal{M} \equiv \langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ be any positive arithmetical interpretation. Then the classical contradiction $F \wedge \neg F$ is true in \mathcal{M} .*

Proof. By Lemma 11 if φ is positive classical then $\varphi(F)$ is true. Moreover, by Definition 14, $\varphi(\neg F)$ is in either $\neg \text{Pr}_{\mathbf{PA}}(\varphi(p))$ if F is the atom p , or $\neg \text{Pr}_{\mathbf{PA}}(\varphi(F) \wedge \text{Con}\mathbf{PA})$ in the most general case. Both formulas are true, and the thesis holds. ■

Proposition 17 *Let F be any arbitrarily complex positive classical formula. Then the instance $\neg(F \wedge \neg F)$ of the classical non-contradiction principle is false in each positive arithmetical interpretation $\mathcal{M} \equiv \langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$.*

Proof. By Proposition 16 we know that $\varphi(F \wedge \neg F)$ is true and not \mathbf{PA} -provable. Then, by definition, $\varphi(\neg(F \wedge \neg F))$ is $\text{Pr}_{\mathbf{PA}}(\varphi(F \wedge \neg F))$ which is false. ■

Proposition 18 *Let F be any arbitrarily complex positive classical formula, such that a non-positive arithmetical interpretation $\mathcal{H} \equiv \langle \mathbf{N}, \mathbf{PA}, \psi \rangle$ exists with $\psi(F)$ true in \mathcal{H} . Then the classical contradiction $F \wedge \neg F$ is true in \mathcal{H} .*

Proof. Take as F any arbitrarily complex positive classical tautology. Then $\psi(F)$ is necessarily true, and the proof is the same as that of Proposition 16. ■

Even if it could seem obvious, an important remark is the following: arithmetical interpretations make true *only a specific proper subclass* of the contradictions of the language, and infinitely many contradictions, both classical and non-classical, result as false. For example $\varphi(\neg F \wedge \neg\neg F)$, $\varphi({}^\circ F \wedge \neg{}^\circ F)$ are false for any F in each arithmetical model.

Proposition 19 *Let F be any arbitrarily complex positive classical formula, such that a non-positive arithmetical interpretation $\mathcal{H} \equiv \langle \mathbf{N}, \mathbf{PA}, \psi \rangle$ exists with $\psi(F)$ true in \mathcal{H} . Then the instance $\neg(F \wedge \neg F)$ of the classical non-contradiction principle is false in \mathcal{H} .*

Proof. See the proof of Proposition 17. ■

The propositions above allow to state the following theorem:

Theorem 9 *Infinitely many arithmetical interpretations exist, both positive and non-positive, that falsify infinitely many instances of the non-contradiction principle $\neg(A \wedge \neg A)$ of classical logic \mathbf{LK} . Moreover, the falsified instances may*

have arbitrarily high grade and include all classical connectives. ■

Corollary 10 *Each fragment \mathbf{U} of classical logic \mathbf{LK} including the non-contradiction principle $\neg(A \wedge \neg A)$ is falsified by suitable arithmetical interpretations, both positive and non-positive. ■*

Now we show that the classical and intuitionistic theorem $B \rightarrow \neg\neg B$, i.e. the *right double negation principle*, is in general falsified by arithmetical interpretations.

Proposition 20 *Let F be any positive classical formula and let $\mathcal{K} \equiv \langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ be any positive arithmetical model. Then the instance $F \rightarrow \neg\neg F$ of the right double negation principle is false in \mathcal{K} .*

Proof. By Lemma 11 if φ is positive classical then $\varphi(F)$ is true. By definition of arithmetical interpretation on positive classical formulas $\varphi(\neg F)$ is in either $\neg \text{Pr}_{\mathbf{PA}}(\varphi(p))$, if F is the atom p , or $\neg \text{Pr}_{\mathbf{PA}}(\varphi(F) \wedge \text{ConPA})$ in the most general case. Both formulas are true and not \mathbf{PA} -provable. Thus $\varphi(\neg(\neg F))$ is $\text{Pr}_{\mathbf{PA}}(\varphi(\neg F))$ which is false. ■

Proposition 21 *Let F be any arbitrarily complex positive classical formula, such that a non-positive arithmetical interpretation $\mathcal{H} \equiv \langle \mathbf{N}, \mathbf{PA}, \psi \rangle$ exists with $\psi(F)$ true in \mathcal{H} . Then the instance $F \rightarrow \neg\neg F$ of the right double negation principle is false in \mathcal{H} .*

Proof. Take as F any positive classical tautology. The proof is similar to that of Proposition 20. ■

Theorem 10 *Each fragment \mathbf{V} of intuitionistic logic \mathbf{LJ} including the right double negation principle $A \rightarrow \neg\neg A$, and possibly excluding each instance of the non-contradiction principle $\neg(A \wedge \neg A)$, is falsified by suitable arithmetical interpretations, both positive and non-positive. Moreover, the falsified proper \mathbf{V} -theorems may have arbitrarily high grade and include all classical connectives. ■*

Finally, it is important to show directly that both *classical negation rule on the left* $\neg - L2$ and *classical contraposition principle* are falsified by arithmetical interpretations.

Proposition 22 *Let $\frac{\Gamma \vdash \Delta, F}{\neg F, \Gamma \vdash \Delta}$ any instance of the classical rule $\neg - L2$, such that F is a positive classical formula. Suppose that for each positive arithmetical interpretation $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$, in the non-trivial cases where $(\Gamma \vdash \Delta)^\varphi$ is false, $(\Gamma \vdash \Delta, F)^\varphi$ is true. Then, the sequent $(\neg F, \Gamma \vdash \Delta)^\varphi$ is false.*

Proof. By definition $\varphi(\neg F)$ is $\neg \text{Pr}_{PA}(\varphi(p))$, if F is the atom p , or $\neg \text{Pr}_{PA}(\varphi(F) \wedge \text{ConPA})$ in the most general case, which are true. Then we have the thesis, since, by assumptions, Δ^φ is false or empty, and Γ^φ is true or empty. ■

Proposition 23 *Assume that F, G are positive classical formulas. Then the contraposition principle instance $(\neg F \rightarrow G) \rightarrow (\neg G \rightarrow \neg\neg F)$ is false in each positive arithmetical interpretation $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$.*

Proof. Recalling the proofs of the propositions above, we have $\varphi(F)$ true, $\varphi(G)$ true, $\varphi(\neg F)$ true, $\varphi(\neg G)$ true, $\varphi(\neg\neg F)$ false. ■

10 Possible developments of paraconsistent proof-theory

The proof-theoretic analysis of **BC**, **CI**, **CIL** presented in the sections from 2 to 6, besides producing some essential tools for the arithmetical semantics, also indicates some possible developments in the investigation, *through a purely syntactic approach*, of Proof-theory and Provability Logic of paraconsistent Arithmetic.

10.1 C-system based Arithmetic and the fundamental conjecture

In [4] and in [14] the Primitive Recursive Arithmetics **PCA** and **PRACI**, respectively based on **BC** and **CI**, have been introduced, and their peculiar paraconsistent Provability Logic has been explored. We deem that the **CI**-based full Arithmetic **PACI**⁶ could be the main system that allows to introduce a new kind of constructive mathematics. We recall that, for each recursively axiomatized system **T**, a **T**-provability predicate $\text{Pr}_{\mathbf{T}}(\cdot)$ can be defined in **PCA** and **PRACI**, such that $\text{Pr}_{\mathbf{T}}(\#B)$ means “the sentence B is **T**-provable”, where $\#B$ is the Gödel-number of B . Indeed, the point is that some interesting relations can be discovered in **PRACI** and **PACI** between *local consistency assertions* of the form ${}^\circ B$ and *global consistency statements* (or *global non-triviality statements*) of the form $\neg \text{Pr}_{\mathbf{T}}(\#B)$. In [14] we have shown that:

$$\vdash \text{Pr}_{\mathbf{CI}}(\#{}^\circ B) \rightarrow \neg \text{Pr}_{\mathbf{CI}}(\#B)$$

is **PRACI**-provable for a suitable class of sentences B 's which have not the form ${}^\circ F$. We call $\text{Pr}_{\mathbf{T}}(\#{}^\circ B) \rightarrow \neg \text{Pr}_{\mathbf{T}}(\#B)$ the *fundamental relation between*

⁶ We recall that: **PCA** and **PRACI** have induction rule on ω with atomic induction formulas, **PACI** is **PRACI** extended by induction rule on ω with arbitrarily complex induction formulas.

local and global consistency for any **CI**-based system **T**. The preliminary results obtained in [4] and in [14] allow to formulate the following *fundamental conjecture*:

Conjecture 1 *It is possible to find a weak **PACI**-extension **W** such that, for a suitable set of sentences B , the fundamental relation for **PACI**, i.e. the sequent $\text{Pr}_{\mathbf{PACI}}(\#^\circ B) \vdash \neg \text{Pr}_{\mathbf{PACI}}(\#B)$, is **W**-provable. ■*

Thus, the problem of proving the non-triviality of **PACI** (i.e a *global* selfreference statement) could be reduced to the provability of suitable *local* consistency assertions – that is, **PACI** *almost* would establish its own non-triviality. In essence, the conjecture is the following: the paraconsistent arithmetical systems based on Logics of Formal Inconsistency have, w.r.t non-triviality, more constructive and efficient proof capabilities than that owned by classical arithmetical systems w.r.t. consistency. If the *fundamental conjecture* is true we could declare a kind of *weakened* Hilbert program for paraconsistent Arithmetic. The substance of the fundamental conjecture was pointed out by Professor Carnielli in 2005.

10.2 *Suggestions for a renewed Hilbert program, from Relevant Arithmetic to LFI based Arithmetic*

Let us discuss what a renewed Hilbert program for paraconsistent Arithmetic can be today. First, if the possibility of a weakened Hilbert program is mentioned, a comparison is necessary with the perspective indicated by Meyer and Mortensen in [20], and also repeated in the Mortensen’s book [21], concerning the same topic. In [20] it is shown that paraconsistent Relevant Arithmetic **R** is absolutely consistent, i.e. non trivial, through a model theoretic argument employing a model with a finite domain. Thus, the authors say that the absolute consistency of **R** is proven by finitistic methods: “[...] *this argument relies only on methods which are finitistic* [...]” ([21] p. 18) and, moreover, “[...] *this goes some way to resurrecting the program of Hilbert of a finitistic demonstration of the (absolute) consistency of mathematics*” ([20] p. 919). We think that such result is very relevant, and represents a substantial step in the development of paraconsistent mathematics. However, in order to better understand it, it should be more clearly explained what is meant for “*finitistic methods*” and in particular what is the *finitistic* character of the mentioned demonstration. Anyway, we note that the result is obtained using induction, since, in general, to establish that any structure is a model for a formal system with infinite theorems, some induction rule or axiom is necessary (if we read the proof in [21] p. 18 the use of induction is implicit but evident). Moreover, it is also evident that the meta-logic employed to demonstrate the properties of **R** and of relevant logic **RQ** is classical logic; in particular, so it must be

the meta-logic providing semantics, i.e. that defining the model theory for \mathbf{R} . Thus, we interpret *finitistic methods accepting some weak instances of induction rule* as *arguments provable in classical Primitive Recursive Arithmetic PRA*. Then, *if we accept the result in its strongest form*, we can suppose that it can be essentially expressed as:

$$\mathbf{PRA} \vdash \text{Non} - \text{Triv}(\mathbf{R})$$

where $\text{Non} - \text{Triv}(\mathbf{R})$ is the \mathbf{PRA} -formula stating the absolute consistency of \mathbf{R} . That is, we suppose that the weakest induction rule, allowing only atomic induction formulas, is needed (even if, as wellknown, \mathbf{PRA} proves induction on boolean combinations of Σ_1 -formulas). Then, an interesting aspect is that classical Arithmetic with the weakest induction rule proves the absolute consistency of an expressive arithmetical system such as \mathbf{R} , with arbitrarily powerful induction rules on ω , i.e. allowing arbitrarily complex induction formulas. Obviously, it must be remarked that the implication of \mathbf{R} is the relevant and not the classical one, and that the positive propositional logic of \mathbf{R} is a proper fragment of classical positive propositional logic. Finally, it is useful to recall that \mathbf{R} is negation consistent too, but for \mathbf{R} absolute consistency and negation consistency are separate notions (see e.g. [21] p. 19), and the discussed result does not mention the negation consistency of \mathbf{R} . A further interesting aspect of the Mayer-Mortensen work is that \mathbf{R} can be extended to an axiomatizable (see [20] p. 920) negation inconsistent system $\mathbf{RM32}$ having a finite model. We remark that being $\mathbf{RM32}$ axiomatizable, a provability predicate $Pr_{\mathbf{RM32}}(\cdot)$ can be defined, so that $\mathbf{RM32}$ -provability can be formalized in \mathbf{PRA} . Thus, a fascinating suggestion arises: adding negation inconsistency in a system that remains non trivial, could reduce the complexity of induction rules required for proving the absolute consistency.

Our proposal of a weakened Hilbert program starting from the properties of \mathbf{LFI} based arithmetical systems, considers a different perspective. We are looking *both for a non triviality proof and for a negation consistency proof of full PACI*, that could be said constructive but not finitistic, in the following sense: the full \mathbf{PACI} induction on ω can be used, but transfinite induction on ordinals up to ε_0 is replaced by *finitistic* statements given by finite propositional combinations $H(\circ B_1, \circ B_2, \dots, \circ B_n)$ of local consistency assertions (such that $\circ B_j$ is not a \mathbf{PACI} bottom particle or theorem and H is not a \mathbf{PACI} bottom particle). Indeed, as wellknown, in principle, a (syntactic) consistency proof of Arithmetic with full induction requires transfinite induction on ordinals greater than ω , up to ε_0 (see, e.g., [23]). Thus, we properly speak of a *weakened* Hilbert program, since *full* induction on ω cannot be said finitistic in the traditional sense. However, it must be remarked that we wish to use proof-theoretic arguments only, and that a real renewal of Hilbert program cannot omit negation consistency. Then, the key point of the research is to explore if [\mathbf{PACI} induction on ω] *plus* [assumptions of the form

$H(^\circ B_1, ^\circ B_2, \dots, ^\circ B_n)$] can replace transfinite induction up to ε_0 . The idea is that local consistency assertions are a kind of formalized metatheory, only requiring propositional language, perhaps the weakest formalized metatheory which is possible, that can however include powerful inference.

The proof of the *fundamental conjecture* would provide a technical corroboration of the above sketched design.

11 Work in progress: interpreting paraconsistent Arithmetic into classical Provability Logic

We have already introduced arithmetical systems based on Logics of Formal Inconsistency **LFIs**': in [4] the **BC**-based paraconsistent Primitive Recursive Arithmetic **PCA** and in [14] the **CI**-based Arithmetics **PRACI** and **PACI**. We will focus on the full **PACI**, that we deem the most expressive among the **C**-system based arithmetical systems. Then, in addition to the possible proof-theoretic developments mentioned in Section 10, a parallel work in progress is the definition of *arithmetical models* $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ for **PACI**, by extending the already defined **CI**-models. Thus, also the arithmetical interpretations of the full predicative case of **CI** will be presented. Various new problems must be faced: for example the interpretation of $^\circ(\cdot)$ will be similar but not identical to that presented in Section 7. We cannot explain here the new devices that the interpretation of **PACI** into Provability Logic of **PA** needs, since a whole new article is necessary. However, some perspectives that the new semantical tools make possible can be sketched.

11.1 Future works: breaking classical equivalences between some classes of reflection principles and consistency

The results already obtained in [4] and [14] show that for Arithmetic **PRACI** having only atomic induction rule, or for some weak extensions admitting only induction formulas with a low complexity ($\exists xA$ or $\forall xB$ with A, B atomic, and so on), a kind of bounded cut elimination property can be established, so that some relevant meta-mathematical properties can be obtained by syntactic tools. For example in [4] it is shown, through syntactic arguments, that classical Diagonal Lemma does not hold in general for **PRACI** and that Gödel theorems for **PRACI** must have a deeply different proof w.r.t. the classical case. However, in order to investigate the relevant specific meta-mathematical features that full **PACI** has w.r.t. classical Arithmetic **PA**, *the effectively available syntactic means do not suffice*. Thus, semantical tools provided by arithmetical interpretations can play a substantial role. In a next work we will

develop the following claim:

*by arithmetical semantics for **PACI**, it can be proven that to the classical equivalences between Π_1 -restricted reflection principles and consistency, and between Σ_1 -restricted reflection principles and $\Sigma_0 - \omega$ -consistency, do not correspond analogous equivalences in the **PACI**-setting.*

This could be an important topic. We deem that the wellknown equivalence between reflection principles and consistency is not intuitive, and strictly depends on the employed classical logic: it is not a real mathematical property of arithmetical theories. Thus, as to **LFI** based arithmetical systems, suitable instances of reflection principle could become a tool for consistency proofs. This point too marks the constructive character of **PACI** w.r.t. **PA**.

More technically, we recall that (see [22] pp. 844-848) the Uniform Reflection Principle for an axiomatizable system \mathbf{T} is $RFN(\mathbf{T}) \equiv \forall x(\text{Pr}_{\mathbf{T}}[B(x)] \longrightarrow B(x))$ ⁷ where B has only x free, and the Local Reflection Principle for \mathbf{T} is $Rfn(\mathbf{T}) \equiv \text{Pr}_{\mathbf{T}}(B) \longrightarrow B$, B closed. In the classical setting, assuming that \mathbf{T} is an axiomatizable extension of **PRA**, the following are equivalent over **PRA** (and then over **PA**) : $\neg \text{Pr}_{\mathbf{T}}(0 = 1)$, $Rfn_{\Pi_1}(\mathbf{T})$, $RFN_{\Pi_1}(\mathbf{T})$, where the principles are restricted to Π_1 -formulas, and $\neg \text{Pr}_{\mathbf{T}}(0 = 1)$ is the consistency statement for \mathbf{T} .

In the **PACI**-setting we must previously say what the Σ_1 -class and the Π_1 -class can be. First, the Σ_0 -class is formed by positive classical propositional combinations of atomic formulas of the form $f = g$, f and g recursive terms (as established in [4] and [14] we employ a language with the only predicate $=$ (\cdot, \cdot) and the names of all recursive functions). The reasons of the exclusion from F of the intensional connectives \neg and $\circ(\cdot)$ is obvious. Then, the class of Σ_1 -sentences of **PACI** includes elements of the form $\exists x_1 \dots \exists x_r F$, $F \in \Sigma_0$. But the negation $\neg F$ with $F \in \Sigma_0$ is neither in Σ_0 nor in Σ_1 . Indeed, as already shown in [4], we remind that, in **PRACI**, not only $\neg f = g$ has an intensional character which makes it completely different from the boolean complement of $f = g$, but also the standard Σ_1 -property $\neg f = g \longrightarrow \text{Pr}_{\mathbf{PRACI}}(\neg f = g)$ does not hold, i.e $\neg f = g$ cannot be considered as **PRACI**-equivalent to a Σ_1 -formula. Thus, on one hand, the Π_1 -class must be splitted at least into two classes of formulas: $\Pi_{1a} \equiv \{\neg \exists x_1 \dots \exists x_r F\}$ and $\Pi_{1b} \equiv \{\forall x_1 \dots \forall x_r F\}$, with $F \in \Sigma_0$; on the other hand, $\forall x \neg(r = s)$ cannot be seen as the universal quantification of a Σ_0 or Σ_1 -formula. Then, we exclude $\forall x_1 \dots \forall x_r \neg F$, $F \in \Sigma_0$, from the Π_1 -class of the **PACI**-setting. After this, by arithmetical semantics of **PACI**, we shall prove that absolute consistency of **PACI** and reflection principle $Rfn_{\Pi_{1a}}(\mathbf{PACI})$ restricted to Π_{1a} -formulas cannot be **PACI**-equivalent.

⁷ As usual, to indicate the provability predicate that preserves the free variables of the possible open formulas at the argument, square brackets are employed, so that $\text{Pr}_{\mathbf{T}}[B(x, y, z, \dots)]$ is a formula preserving the free variables x, y, z, \dots

As to the ω -consistency, we recall that in the classical setting, if \mathbf{T} is an axiomatizable extension of \mathbf{PRA} , ω -consistency of \mathbf{T} can be formalized as (see [22] p. 853) $\text{Pr}_{\mathbf{T}}(\exists x B(x)) \longrightarrow \exists x \neg \text{Pr}_{\mathbf{T}}[\neg B(x)]$, where B has only x free. We write Σ_k - ω -consistency for the restriction of the schema to formulas $B \in \Sigma_k$. We have that, over \mathbf{PRA} (and then over \mathbf{PA}), the Σ_1 -restricted reflection principle $\text{Rfn}_{\Sigma_1}(\mathbf{T})$ and Σ_0 - ω -consistency of \mathbf{T} are equivalent. Conversely, we shall prove by arithmetical semantics of \mathbf{PACI} that such equivalence does not hold in \mathbf{LFI} based paraconsistent Arithmetic, that is $\text{Rfn}_{\Sigma_1}(\mathbf{PACI})$ and Σ_0 - ω -consistency of \mathbf{PACI} are not equivalent over \mathbf{PACI} .

11.2 Future works: looking for a conjectural reasoning naturally arising from \mathbf{LFI} based Arithmetic

A further different topic that can be investigated concerns the conjectural reasoning that can be expressed inside \mathbf{PACI} , that should represent a peculiar property of \mathbf{LFI} based arithmetical systems w.r.t. classical Arithmetic. That is, we will explore the following problem: *what are the contradictions $F \wedge \neg F$ of the \mathbf{PA} -language (which is included in the \mathbf{PACI} -language) that result as true in the arithmetical models of \mathbf{PACI} ?* This is a relevant question, since we could see such sentences as *constructive conjectures* (and then neither probabilistic nor possibilistic) having a mathematical interest, in particular in the cases where F is \mathbf{PA} -consistent, or both F and $\neg F$ are \mathbf{PA} -consistent. Besides the class of (arithmetical) *constructive conjectures*, also the class of (arithmetical) *paradoxical assertions* could be interesting: a *paradoxical assertion* is each non-atomic sentence A of the \mathbf{PA} -language which is a proper \mathbf{PA} -theorem and is falsified by any arithmetical model $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ of \mathbf{PACI} . The class of paradoxical assertions would be not empty. Assuming that infinitely many instances $\neg(F \wedge \neg F)$ of the non-contradiction principle in the \mathbf{PA} -language are falsified by suitable arithmetical models of \mathbf{PACI} , as for the \mathbf{CI} -language happens (Section 9), if G is a proper \mathbf{PA} -theorem, then $G \wedge \neg(F \wedge \neg F)$ is a paradoxical assertion. Thus, we have the (apparent) **arithmetical semantics paradox**: *provability logic of \mathbf{PA} falsifies infinitely many proper \mathbf{PA} -theorems.*

Obviously, paradoxical assertions of the form $G \wedge \neg(F \wedge \neg F)$ would not be really relevant from a mathematical point of view. An open research problem is to find both constructive conjectures and paradoxical assertions that can be mathematically relevant.

12 Acknowledgements

The author wishes to thank Professor Walter A. Carnielli for many helpful discussions, important suggestions, and his constant encouragements to the author's researches on paraconsistent Provability Logic and paraconsistent Arithmetic. The idea of a possible weakened Hilbert's program founded on Logics of Formal Inconsistency was firstly remarked by professor Carnielli in a private communication to the author.

The author wishes also to thank the anonymous referees for technical advices, useful suggestions for the presentation of the possible developments in the direction of meta-mathematics of paraconsistent Arithmetic, and for the encouragement to explore a more systematic characterization of the notion of constructivity in the paraconsistent setting.

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