Introduzione di una logica paraconsistente *costruttiva* e dei sistemi aritmetici su essa basati

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Paolo Gentilini*

**ANSAS Liguria ,*
e
Istituto di Matematica Applicata e Tecnologie Informatiche del Consiglio Nazionale delle Ricerche (IMATI-CNR)
Genova, Italy

gentilini@ima.ge.cnr.it www.gentilini.ge.it

Constructive Paraconsistent Logic: goals and motivations

To introduce a Paraconsistent Arithmetic which can properly fit in the main stream of Proof-Theory and Provability Logic of Classical Arithmetic and Intuitionistic Arithmetic (and, possibly, Classical and Intuitionistic Analysis). The idea is that paraconsistent reasoning, intuitionistic reasoning and classical reasoning can be compared as different ways to think about the *same elementary mathematical objects*.

Basic program

- Starting from the paraconsistent logic framework of Logic of Formal Inconsistency (**LFI**) (Carnielli et al. 2002) the **C**-systems **bC**, **Ci**, **Cil** are selected [LFI's systems can possibly support contradictions $A\wedge\neg A$ without trivializing, but they do non prove contradictions i.e. they **are not** dialectical logics]

LFIs' extend the classical language through a monadic propositional connective °(.), such that the intended meaning of °*B* is

"*B* is consistent" that is "<<*B* and not *B*>> does not hold",

Formulas of the form °*A* are called *local consistency assertions*.

Basic program

- The sequent versions **BC**, **CI**, **CIL** are provided, and the cut elimination property is proven (Gentilini 2005, 2009)

- An *arithmetical semantics* is given which interprets **C**-system formulas into Provability Logic sentences of classical Arithmetic **PA**: thus, it links the notion of truth to the notion of provability inside a classical environment. In particular, the intensional paraconsistent negation is in general interpreted *as a non-provability condition*, i.e. $\varphi(\neg B) \equiv \neg\, Pr_{\mathbf{PA}}(\varphi(B))$

- **BC**, **CI**, **CIL admit arithmetical models** (Gentilini, 2009, 2011). **Arithmetical semantics selects a class of some contradictions** $B\wedge\neg B$ that are true in the arithmetical models. We call them *constructive contradictions*.

Extended program

- To define a **formal notion of constructivity** for **LFI-**paraconsistent logic (Gentilini 2011). It is shown that **CI** is a *constructive* paraconsistent system.

-To explore a kind of Weakened Hilbert Program referred to **CI-**based Primitive Recursive Arithmetic **PRACI** (with atomic induction up to ω) and **CI-**based Arithmetic **PACI** (with induction up to ω on arbitrary formulas).

CONJECTURE

[**PACI** *induction on* ω] ***plus*** [“*finitistic*” *statements given by finite propositional combinations* $H(°B_1,°B_2,\ldots,°B_n)$ *of local consistency assertions*] can prove the **absolute consistency** and the **negation consistency** of **PACI** and **PA**.
(i.e a weak “finitistic” extension of **PACI** proves the **absolute consistency** of **PACI**)

MAIN CONJECTURE (MC)

a)There exists a **finite** set $\Omega \equiv \{B_j \wedge \neg B_j\}$ of constructive contradictions in the arithmetical language such that **CI**-based Elementary Arithmetic **EAI** *plus* Ω (without induction) is paraconsistent and proves **PRACI**-induction [assuming that no system among **EAI** *plus* B_j **,** **EAI** *plus* $\neg B_j$ proves any induction]

b)There exists a **finite** set $\Omega \equiv \{F_j \wedge \neg F_j\}$ of constructive contradictions in the arithmetical language such that **PRACI** *plus* Ω is paraconsistent and proves both the consistency and the negation-consistency of **PA** and all the **PA**-theorems of complexity $\Pi 1$ [assuming that no system among **PRACI** *plus* F_j**,** **PRACI** *plus* $\neg F_j$ proves any non atomic induction]

If **MC** holds, then very strong arguments against Church's Thesis can be shown

Selected References

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(preprint at http://www.gentilini.ge.it)

P. Gentilini, ‘Paraconsistent Arithmetic with a Local Consistency Operator and Global Selfreference’, *Electronic Notes in Theoretical Computer Science* (ENTCS), 169, 2007, 73-86.

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Sequent C-systems BC, CI, CIL : propositional part

Axioms: $A \vdash A$

Positive propositional logical rules (X, Γ,…multisets):

$$\frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge\text{–L} \qquad \frac{B, \Gamma \vdash \Delta}{B \wedge A, \Gamma \vdash \Delta} \wedge\text{–L}$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Lambda \vdash \mathrm{X}, B}{\Gamma, \Lambda \vdash \Delta, \mathrm{X}, A \wedge B} \wedge\text{–R}$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \vee\text{–R} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, B \vee A} \vee\text{–R}$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Lambda \vdash \mathrm{X}}{A \vee B, \Gamma, \Lambda \vdash \Delta, \mathrm{X}} \vee\text{–L}$$

$$\frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B} \rightarrow\text{- R} \qquad \frac{\Gamma \vdash \Delta, A \qquad B, \Lambda \vdash \mathrm{X}}{A \rightarrow B, \Gamma, \Lambda \vdash \Delta, \mathrm{X}} \rightarrow\text{- L}$$

The Paraconsistent sequent systems CI and CIL

CI *Negation rules:*

$$\frac{A, \Gamma \mid\!\!\!— \Delta}{\neg\neg A, \Gamma \mid\!\!\!— \Delta} \ \neg - \mathrm{L1} \qquad \frac{\circ A, \Gamma \mid\!\!\!— \Delta, A}{\circ A, \ \neg A, \Gamma \mid\!\!\!— \Delta} \ \neg - \mathrm{L3}$$

$$\frac{\Gamma \mid\!\!\!— \Delta, \circ A}{\neg\circ A, \Gamma \mid\!\!\!— \Delta} \ \neg - \mathrm{L4} \qquad \frac{A, \Gamma \mid\!\!\!— \Delta}{\Gamma \mid\!\!\!— \Delta, \neg A} \ \neg - \mathrm{R}$$

The formula **°*A*** in the ¬-L3 premise is the ***constraint formula*** of the rule

Introduction rule for °(.) in **CI** :

$$\frac{A \wedge \neg A, \Gamma \mid\!\!\!— \Delta}{\Gamma \mid\!\!\!— \Delta, \circ A} \qquad \mathrm{RCi}$$

The system **CIL** is the extension of **CI** through the following rule *L*5:

$$\frac{\Gamma \mid\!\!\!— \Delta, A \wedge \neg A}{\neg (A \wedge \neg A) \Gamma \mid\!\!\!— \Delta} \ \neg - \mathrm{L5}$$

(*A*∧¬*A*) in the *L*5 premise is the ***constraint formula*** of the rule

The Paraconsistent sequent systems CI and CIL

BC is **CI** *minus* [*RCi*, ¬-L4]

Structural rules:

Weakening rules:

$$\frac{\Gamma \mid\!\!-\!\!- \Delta}{\Gamma \mid\!\!-\!\!- \Delta, A}\ W-R \qquad \frac{\Gamma \mid\!\!-\!\!- \Delta}{A, \Gamma \mid\!\!-\!\!- \Delta}\ W\text{-}L$$

Cut rule:

$$\frac{\Gamma \mid\!\!-\!\!- \Delta, A \qquad A, \Lambda \mid\!\!-\!\!- \mathrm{X}}{\Gamma, \Lambda \mid\!\!-\!\!- \Delta, \mathrm{X}}\ \text{Cut}$$

Contraction rules:

$$\frac{\Gamma \mid\!\!-\!\!- \Delta, A, A}{\Gamma \mid\!\!-\!\!- \Delta, A}\ C-R \qquad \frac{A, A, \Gamma \mid\!\!-\!\!- \Delta}{A, \Gamma \mid\!\!-\!\!- \Delta}\ C\text{-}L$$

DIAGONAL NEGATION RULES

We will discuss about constructivity only for paraconsistent systems endowed by *diagonal negation rules*:

DEFINITION *A sequent rule introducing negation is* **diagonal** *if the auxiliary formula is at the left (right) side of the premise and the principal formula is at the right (left) side of the conclusion. We call left (right) diagonal negation rules those with the principal formula at the left (right) side of the conclusion.*

A diagonal negation rule **moves the formula**, changing the sequent side at which it occurs.

$$\frac{A, \Gamma \mid\!\!-\!\!- \Delta}{\neg\neg A, \Gamma \mid\!\!-\!\!- \Delta}\ \neg - \mathrm{L1} \qquad \text{is NOT diagonal}$$

We will focus on diagonal negation rules. A relevant class of diagonal negation rules is given by rules **with the constraint formula property**:

Negation rules with the constraint formula property

DEFINITION

i) ***A left diagonal negation*** rule of a system **U** has **the constraint formula property** if it has the following form:

$$\frac{\Phi, \Gamma \mid\!\!-\!\!- \Delta, A}{\neg A, \Phi, \Gamma \mid\!\!-\!\!- \Delta} \qquad H$$

where exactly one of the following cases holds:

a) the set **Φ** is empty, and *A* is a fixed formula schema Λ;

b) no constraints are posed on *A* and the set **Φ** is a singleton including a fixed formula schema Σ, such that infinitely many instances of Σ are neither theorems nor bottom particles of **U**. In each *H*-instance, the **Σ**-instance δ is called **the constraint formula** of the rule instance.

Negation rules with the constraint formula property

ii) ***A right diagonal negation rule*** of a system **U** has **the constraint formula property** if it has the following form:

$$\frac{B, \Gamma \mid\!\!-\!\!- \Delta, \Psi}{\Gamma \mid\!\!-\!\!- \Delta, \Psi, \neg B} \quad R$$

where exactly one of the following cases holds:

c) the set Ψ is empty and B is a fixed formula schema Θ;

d) no constraints are posed on B and the set Ψ is a singleton including a fixed formula schema Π such that infinitely many instances of Π are neither theorems nor bottom particles of **U**. In each R-instance, the Π instance σ is called **the constraint formula** of the rule instance.

Assumption

Paraconsistent systems are characterized by *left* diagonal negation rules having proper constraints on the **left side** (for example the ¬-L3 rule of **CI**) and (general)-intuitionistic systems are characterized by *right* diagonal negation rules having proper constraints on the **right** side (for example the ¬-R rule of **LJ**)

Then, it is relevant to formalize *left diagonal rules constrained on the left*, and *right diagonal rules constrained on the right*.

BC, CI, CIL provide examples of left diagonal rules with the constraint formula property:

$$\frac{\circ A, \Gamma \mid\!\!-\!\!- \Delta, A}{\circ A, \neg A, \Gamma \mid\!\!-\!\!- \Delta} \neg - \mathrm{L3} \qquad \frac{\Gamma \mid\!\!-\!\!- \Delta, \circ A}{\neg \circ A, \Gamma \mid\!\!-\!\!- \Delta} \neg - \mathrm{L4}$$

$$\frac{\Gamma \mid\!\!-\!\!- \Delta, A \wedge \neg A}{\neg (A \wedge \neg A)\, \Gamma \mid\!\!-\!\!- \Delta} \neg - \mathrm{L5}$$

Relationship between intuitionism and paraconsistency: background

1 Brunner-Carnielli duality (connective duality)

Through a dualization operation (.)* on connectives : $(A\wedge B)^* \equiv A\vee B$, $(A\vee B)^* \equiv A\wedge B$ $(A\rightarrow B)^* \equiv B^* - A^*$, $(\neg A)^* \equiv \top - A^*$ [where " $-$ " is the (intensional) *pseudo-difference* or *coimplication* connective] from an intuitionistic propositional system S, a dual system S* can be defined which is paraconsistent.

- A. B. M. Brunner, W. A. Carnielli, "Anti-intuitionism and paraconsistency", Journal of Applied Logics 3(1), 2005, 161-184.

2 Urbas-Aoyama duality (sequent duality)

The paraconsistent systems **LDJ** of Urbas and **DI** of Aoyama are (essentially) defined through the converse of the condition for Intuitionistic Logic **LJ**: in **DI**, **LDJ** each sequent in a proof has at most a singleton as antecedent.

- I. Urbas, "Dual-Intutionistic Logic", Notre Dame Journal of Formal Logic, 37, 3, 1996, 440-451.
- H. Aoyama, "LK, LJ, Dual Intuitionistic Logic, and Quantum Logic", Notre Dame Journal of Formal Logic, 45, 4, 2004, 193-213.

Proof–Theoretic formalization of paraconsistency/intuitionism duality

We formally introduce the idea that a ***constructive paraconsistent system*** **U** should have an ***intuitionistic reference system*** **V** from which the negation rules of **U** can be obtained through canonical manipulations.

Essentially, we expect a **kind of antisymmetry relation** between premises and conclusions of the diagonal negation rules of the two systems, linking *left diagonal negations* of **U** to *right diagonal negations* of **V**.

This imposes to enlarge the class of intuitionistic systems to that of **pseudo-intuitionistic systems**:

DEFINITION *Let* **V** *be a sequent system in the language of* Logic of Formal Inconsistency, *including the positive rules of classical logic* **LK** *and endowed by* ***diagonal*** *negation rules. Then* **V** *is a* **pseudo-intuitionistic system** *if* it has cut-elimination *and, moreover, infinitely many instances of the* excluded middle principle $B \vee \neg B$ *and of the* left double negation principle $\neg\neg B \rightarrow B$ *are* ***not*** **V**-*provable.*

Introducing *negation-dual* rules

DEFINITION Let H be a left diagonal negation rule of a sequent system **U** in the language of Logic of Formal Inconsistency. Then **the negation-dual rule** E **of** H is a right diagonal negation rule so defined:

i) if H has not the constraint formula property, i.e it has the form:

$$\frac{\Gamma \mid\!\!-\!\!- \Delta, A}{\neg A, \Gamma \mid\!\!-\!\!- \Delta} \quad H$$

then E has the form

$$\frac{B, \Omega \mid\!\!-\!\!- \Pi}{\Omega \mid\!\!-\!\!- \Pi, \neg B} \quad E$$

Introducing *negation-dual* rules

ii) **If** *H* has the constraint formula property, i.e it has the form:

$$\frac{\Phi, \Gamma \mid\!\!-\; \Delta, A}{\neg A, \Phi, \Gamma \mid\!\!-\; \Delta} \qquad H$$

where Φ and *A* are as established above, **then**:

a) if Φ is empty and *A* is the fixed formula schema Λ then *E* has the form:

$$\frac{\neg\Lambda, \Omega \mid\!\!-\; \Pi}{\Omega \mid\!\!-\; \Pi, \neg\neg\Lambda} \qquad E$$

b) if Φ is not empty then *E* has the form:

$$\frac{B, \Omega \mid\!\!-\; \Pi, \neg\Sigma}{\Omega \mid\!\!-\; \Pi, \neg\Sigma, \neg B} \qquad E$$

where Σ is the formula schema included in the singleton Φ.

Thus, we apply an antisymmetry operation to the constraint formula of *H*. As evident, *E* is a right diagonal negation rule with the constraint formula property.

The negation-dual rule of a right diagonal negation rule is obtained analogously

EXAMPLE

PROPOSITION The negation-dual rules of the rules : ¬- L3, ¬-L4, ¬-L5 of the systems **BC**, **CI**, **CIL** are the following:

$$\frac{A, \Omega \mid\!\!\text{---}\ \Pi, \neg^{\circ} A}{\Omega \mid\!\!\text{---}\ \Pi, \neg^{\circ} A, \neg A}\ \neg - \text{R3} \qquad \frac{\neg^{\circ} A, \Omega \mid\!\!\text{---}\ \Pi,}{\Omega \mid\!\!\text{---}\ \Pi, \neg\neg^{\circ} A}\ \neg - \text{R4}$$

$$\frac{\neg (A \wedge \neg A), \Omega \mid\!\!\text{---}\ \Pi}{\Omega \mid\!\!\text{---}\ \Pi, \neg\neg (A \wedge \neg A)}\ \neg - \text{R5}$$

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$$\frac{{}^{\circ}A, \Gamma \mid\!\!\text{---}\ \Delta, A}{{}^{\circ}A,\ \neg A, \Gamma \mid\!\!\text{---}\ \Delta}\ \neg - \text{L3} \qquad \frac{\Gamma \mid\!\!\text{---}\ \Delta, {}^{\circ}A}{\neg^{\circ}A, \Gamma \mid\!\!\text{---}\ \Delta}\ \neg - \text{L4}$$

$$\frac{\Gamma \mid\!\!\text{---}\ \Delta, A \wedge \neg A}{\neg (A \wedge \neg A)\ \Gamma \mid\!\!\text{---}\ \Delta}\ \neg - \text{L5}$$

The *negation-dual* system V of a paraconsistent system U

DEFINITION *Let* **U** *be a paraconsistent sequent system in the language of Logic of Formal Inconsistency, such that applying to* **U** *the standard translation t with* $t(°A) \equiv \neg (A \wedge \neg A)$ *a formulation of a subsystem of Classical Logic* **LK** *is obtained, and such that* **U** *has both left anf right diagonal negation rules. Then the* ***negation-dual system*** **nd -U** *of* **U** *is* [**structural and positive rules of LK**] ***plus*** [**negation-dual rules of the diagonal negation rules of U**]:

Pseudo-constructive paraconsistent systems

DEFINITION *Let* **U** *be a paraconsistent sequent system in the language of Logic of Formal Inconsistency, such that the negation-dual system* **nd –U** *can be defined. Then we say that* **U** *is* **pseudo-constructive** if **U** *has cut-elimination and* **nd –U** *is pseudo-intuitionistic.*

Then, in general, a bit of work is needed in order to establish the pseudo-constructivity of a system.

The interesting point is to investigate the constructivity of paraconsistent systems whose diagonal negation rules have **the constraint formula property**.

Declarative constructivity

We note that the **constraint formula** of a diagonal negation rule of a system **U** in a sense *declares* (and measures) the **distance** between **U** and Classical Logic **LK. Thus:**

DEFINITION *Let* **U** *be a pseudo-constructive paraconsistent system. Then,* **U** *is* **declaratively constructive** *if it has* ***left diagonal negation rules*** *with* ***the constraint formula property***.

THEOREM The negation-dual system **nd -U** of **CI** is given by:
[structural and positive rules of **LK**] ***plus***

$$\frac{A, \Omega \mid\!\!\text{—}\; \Pi, \neg^{\circ} A}{\Omega \mid\!\!\text{—}\; \Pi, \neg^{\circ} A, \neg A}\ \neg - \text{R3} \qquad \frac{\neg^{\circ} A, \Omega \mid\!\!\text{—}\; \Pi,}{\Omega \mid\!\!\text{—}\; \Pi, \neg\neg^{\circ} A}\ \neg - \text{R4}$$

$$\frac{\Gamma \mid\!\!\text{—}\; \Delta, A}{\neg A, \Gamma \mid\!\!\text{—}\; \Delta}\ \neg - \text{L}$$

and it is a pseudo-intuitionistic system.

COROLLARY: CI is a declaratively constructive paraconsistent system.

Canonical constructivity: negation as unprovability condition

We recall that a seminal idea in constructive mathematics is that the negation of *A* must express an unprovability condition of *A*.

ARITHMETICAL SEMANTICS for paraconsistent **C**-systems can formalize ***negation as unprovability condition*** (NUC).

DEFINITION Let **U** be a paraconsistent sequent system in the language of Logic of Formal Inconsistency. An **arithmetical interpretation** of the **U**-language is a triple ⟨**N**, **PA**, φ ⟩ where:

- **N** is the standard model of Classical Arithmetic **PA**;
- φ is an application such that atomic propositional formulas p_i, p_r,.. are sent into **PA**-formulas of the forms $Con\mathbf{W}_j$, $\neg Con\mathbf{W}_k$,… i.e $\neg\exists x Prov_{\mathbf{W}j}(x,\#0=1)$, $\exists x\, Prov_{\mathbf{W}j}(x,\#0=1)$,… $\mathbf{W}_j$, $\mathbf{W}_{j\ldots}$ consistent axiomatizable extensions of **PA**;
- for each compound formula *B*, φ (*B*) is a formula of **PA**-Provability Logic.

B **is true in** ⟨**N**, **PA**, φ ⟩ if **N /=** φ (*B*) in the standard sense.

⟨**N**, **PA**, φ ⟩ is an **arithmetical model** of **U** if **N /=** φ (*A*) for each **U**-theorem *A*.

Negation as unprovability condition

In particular we have that, for compound formulas:

If *B* is not a positive classical formula then:

$\varphi\ (°B) \equiv Con\mathbf{PA}$

$\varphi\ (\neg \boldsymbol{B}) \equiv \quad \neg\ Pr_{\mathbf{PA}}(\varphi(\mathrm{B}))$ if $\varphi\ (B)$ is **N**-false

$Pr_{\mathbf{PA}}(\varphi(\mathrm{B}))$ if $\varphi(B)$ is not **PA**-provable and **N**-true

$\neg\ Pr_{\mathbf{PA}}(\varphi(\mathrm{B}))$ if $\varphi\ (B)$ is **PA**-provable

THEOREM BC, CI, CIL admit arithmetical models

DEFINITION *Let* **U** *be a declaratively constructive paraconsistent system. Then* **U** *is* **canonically constructive** *if it has the following* **formal NUC-property**: **U** *admits an arithmetical model* $\langle \mathbf{N}, \mathbf{PA}, \varphi \rangle$ *such that for infinitely many non-atomic formulas B of* **U** *we have* $\varphi\ (\neg \boldsymbol{B}) \equiv \neg\ Pr_{\mathbf{PA}}(\varphi(\mathrm{B}))$.

PROPOSITION: CI is a canonically constructive paraconsistent system.