A new semantics for predicate logic

Characterizing First Order Logic through constructive models for Type Theory

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OUTLINE

Inferential semantics for Type Theory LKw

LKw: a Sequent formulation of Type Theory
Abstract Deduction Structures (ADS) in LKw-trees
Comprehension rules and Comprehension ADS
Inferential Algebra Inf A
Interpretations and semantical frames w.r.t. Inf A and completeness

LKw as semantical environment for first order logic

<u>Searching for a semantics of first oder proofs:</u>
- the Inf A sub-algebra characterizing standard first order logic LK1
- bounded complexity of models of LK1-formulas and completeness

Work in progress

The Inf A sub-algebras characterizing standard finite order logics (second order and so on)
Semantics of proofs for finite order logics

(Formal)Inferential semantics: general aims

Formal Inferential semantics for higher order logic (HOL) is a research program that aims to develop the following perspectives:

To give a semantics for HOL having a constructive character

To allow a formal semantical distinction between meaning and truth value of a sentence

To produce the tools for a constructive semantics of proofs

Essentially, typed formulas are interpreted into an algebra of sequent-trees, and suitable effective reductions between trees correspond to the logical connectives of the interpreted formula

Seminal idea: **to interpret formulas (statements, sentences,…) of predicate logic into quantification acts (cognitive level) i.e. into quantification rules for ∃ and ∀ (logical level)**

References

Main specific reference

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Preliminary specific references

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Language of Type Theory à la Church: notations

(<u>We assume as known the language of λ-calculus</u>)

Logical constants:

1) *propositional logical constants*:

negation $\neg_{o\to o}$ disjunction $\vee_{o\to(o\to o)}$

conjunction $\wedge_{o\to(o\to o)}$

implication $\supset_{o\to(o\to o)}$

syntactic symbol for "true sentence" $\top$ and

syntactic symbol for "false sentence" $\bot$, both of type *o* .

If A_o , B_o are arbitrary formulas of type *o* the following writings are used:

$\neg A$ for $\neg_{o\to o} A_o$

$A \vee B$ for $(\vee_{o\to(o\to o)} A_o)B_o$

$A \wedge B$ for $(\wedge_{o\to(o\to o)} A_o) B_o$

$A \supset B$ for $(\supset_{o\to(o\to o)} A_o) B_o$

$A \leftrightarrow B$ for $(A \supset B) \wedge (B \supset A)$ (all the resulting formulas have type *o*)

Language of Type Theory

2) *quantifier logical constants*: for each type α we have the following contants:

existential constant $\exists_{(\alpha\to o)\to o}$

universal constant $\forall_{(\alpha\to o)\to o}$

For each arbitrary B_o the following writings are used:

$\exists x_\alpha B$ for $\exists_{(\alpha\to o)\to o}\lambda x_\alpha B_o$

$\forall x_\alpha B$ for $\forall_{(\alpha\to o)\to o}\lambda x_\alpha B_o$

3) *Equality constants*: for each type α we have the equality constant

$=_{\alpha\to(\alpha\to o)}$

It is written $A_\beta =_\beta B_\beta$ for $(=_{\beta\to(\beta\to o)} A_\beta)B_\beta$

$[C_\alpha/M_\alpha]B_\beta$ is the result of the uniform replacement of each occurrence of the term M_α in B_β with C_α

The sequent system LKω

[Ω,Δ,Γ, Π,Θ, ... meta-expressions **for sets of *o*-typed formulas** and *A,B,C,D,...* for isolated formulas. Ω, Δ stands for $\Omega \cup \Delta$]

0) *Logical axioms*:

01): $A \vdash A$ A atomic formula

02) $\vdash \top$ $\bot \vdash$

1) *Logical rules*:

Propositional logical rules:

$$\frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \;\wedge\text{–L} \qquad \frac{B, \Gamma \vdash \Delta}{B \wedge A, \Gamma \vdash \Delta} \;\wedge\text{–L}$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Lambda \vdash \mathrm{X}, B}{\Gamma, \Lambda \vdash \Delta, \mathrm{X}, A \wedge B} \;\wedge\text{–R}$$

The sequent system LKω : *Propositional logical rules*

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\ \vee\text{–R} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, B \vee A}\ \vee\text{–R}$$

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Lambda \vdash \mathrm{X}}{A \vee B, \Gamma, \Lambda \vdash \Delta, \mathrm{X}}\ \vee\text{–L}$$

$$\frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}\ \supset\text{- R} \qquad \frac{\Gamma \vdash \Delta, A \qquad B, \Lambda \vdash \mathrm{X}}{A \supset B, \Gamma, \Lambda \vdash \Delta, \mathrm{X}}\ \supset\text{- L}$$

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ \neg\text{– L} \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ \neg\text{ - R}$$

The sequent system LKω: 1ii) *Quantifier logical rules*:

$$\frac{[t\alpha/x_\alpha]A,\ \Gamma \mid\!\!\text{—}\ \Delta}{\forall x_\alpha A, \Gamma \mid\!\!\text{—}\ \Delta}\ \forall - \mathrm{L} \qquad \frac{\Gamma \mid\!\!\text{—}\ \Delta, [b\alpha/x_\alpha]A}{\Gamma \mid\!\!\text{—}\ \Delta, \forall x_\alpha A}\ \forall - \mathrm{R}$$

$$\frac{[b\alpha/x_\alpha]A,\ \Gamma \mid\!\!\text{—}\ \Delta}{\exists x_\alpha A, \Gamma \mid\!\!\text{—}\ \Delta}\ \exists - \mathrm{L} \qquad \frac{\Gamma \mid\!\!\text{—}\ \Delta, [t_\alpha/x_\alpha]A}{\Gamma \mid\!\!\text{—}\ \Delta, \exists x_\alpha A}\ \exists - \mathrm{R}$$

t arbitrary term, *b* free variable which does not occur in Γ, Δ. Moreover, *t* may be not fully quantified while *b* must be uniformly replaced by *x*

1iii) λ-rule:

$$\frac{\Gamma^* \ /\text{---}\ \Delta^*}{\Gamma\ /\text{---}\ \Delta}\quad \lambda$$

where the sets Γ and Γ*and the sets Δ and Δ* differ only in that zero or more formulas in them are replaced by some formulas to which they are *β*-convertible. *β*-reduction may work either upwards or downwards.

The sequent system LKω : 2) Structural rules:

Weakening rules: $\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}$ $W-R$ $\dfrac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}$ $W\text{-}L$

Cut rule: $\dfrac{\Gamma \vdash \Delta, A \qquad A, \Lambda \vdash X}{\Gamma, \Lambda \vdash \Delta, X}$ Cut

3) *Typed equality axioms*:

3i) $\vdash t_\alpha =_\alpha t_\alpha$

t_α arbitrary

3ii) $\vdash (A_o =_o B_o) \leftrightarrow (A_o \leftrightarrow B_o)$

3iii) $t_\alpha =_\alpha f_\alpha ,\ q_\alpha =_\alpha r_\alpha ,\ f_\alpha =_\alpha r_\alpha \vdash t_\alpha =_\alpha q_\alpha$

3iv) $t_\alpha =_\alpha f_\alpha \vdash A_{\alpha\to o}\, t_\alpha =_o A_{\alpha\to o}\, f_\alpha$

3v) *Extensionality axioms* :

$\forall x_\beta (F_{\beta\to\alpha} x_\beta =_\alpha G_{\beta\to\alpha} x_\beta) \vdash (F_{\beta\to\alpha} =_{\beta\to\alpha} G_{\beta\to\alpha})$ $F_{\beta\to\alpha}$, $G_{\beta\to\alpha}$ arbitrary

Complexity measures on types

Two measures on types are considered:

Order of a type: $ord(o) = 1$ $ord(i)=0$
let the *condensed writing* of τ be $\tau_1 \rightarrow \tau_2 \rightarrow \ldots \tau_m \rightarrow a$, a atomic type;
then:
$ord(\tau) = \max\{ord(\tau_1), \ldots, ord(\tau_m)\} + 1.$

Heigth of a type: $h(i)=0$ $h(o)=1$
$h(\alpha \rightarrow \beta) = \max\{h(\alpha), h(\beta)\} + 1.$

Comprehension Axiom and Comprehension Rules

It is well known the centrality of **Comprehension axiom** in higher order logic

<u>Relational writing</u> (in standard predicate logic language) of the schema :

$$\forall X_{\gamma 1}^{1}\ldots\forall X_{\gamma m}^{m}\exists \boldsymbol{Y}_{\boldsymbol{\beta}}\forall Z_{\alpha 1}^{1}\ldots\forall Z_{\alpha n}^{n}(\boldsymbol{Y}_{\boldsymbol{\beta}}(Z_{\alpha 1}^{1},\ldots,Z_{\alpha n}^{n})\leftrightarrow \varphi)$$

where φ is an arbitrary formula having variables included in the set $\{ X_{\gamma 1}^{1}\ldots\forall X_{\gamma m}^{m}, Z_{\alpha 1}^{1},\ldots,Z_{\alpha n}^{n} \}$, so that Y_β is not a variable of φ, $\beta = [\alpha_1,\ldots,\alpha_n, o]$

<u>Functional writing</u>, in the $\mathbf{LK}_\omega$-language:

$$\exists f_{\alpha 1\rightarrow\ldots\rightarrow\alpha n\rightarrow\beta}\forall x_{\alpha 1}^{1}\ldots\forall x_{\alpha n}^{n}(f_{\alpha 1\rightarrow\ldots\rightarrow\alpha n\rightarrow\beta}\; x_{\alpha 1}^{1}\ldots x_{\alpha n}^{n} =_\beta \mathrm{B}_\beta)$$

B_β arbitrary β-typed formula,

$f_{\alpha 1\rightarrow\ldots\rightarrow\alpha n\rightarrow\beta}$ bound variable which cannot coincide with any free variable in B_β

Comprehension rules

We also call ***Comprehension rules*** (briefly ***Comp-rules***) the ∃-R, ∀-L rules acting *on arbitrary formulas of arbitrary type*:

$$\frac{[t_\alpha/x_\alpha]A,\Gamma\vdash\Delta}{\forall x_\alpha A,\Gamma\vdash\Delta}\ \forall-\mathrm{L}\qquad\qquad \frac{\Gamma\vdash\Delta,[t_\alpha/x_\alpha]A}{\Gamma\vdash\Delta,\exists x_\alpha A}\ \exists-\mathrm{R}$$

we remark that the relevant part of higher order inference is expressed by them

It is well known that: :

in a sequent version of type theory, Comprehension Axioms are fully expressed by a definition of ∃-R, ∀-L *rules such that for any type* α *arbitrary formulas* t_α *can be quantified.*

Seminal idea (second version): **to interpret formulas (statements, sentences,…) of predicate logic [and of full higher order predicate logic] into sequences of comprehension rules**

Abstract Deduction Structures (ADS) in LK$_\omega$ -trees

Let P be an **LK$_\omega$**-proof tree. A *Comprehension abstract deduction structure linked to P* (*Comp ADS* in P) is any finite sequence $\mathcal{T} \equiv \{A^j{}_{\beta j}\}$, $j=1,...,r$, $r \geq 2$, of formula occurrences in P, β_j arbitrary types, built as follows:

J) a set **R** of Comp-rule occurrences $\{\mathcal{R}_1,\ldots,\mathcal{R}_h\}$, $h \geq 1$ has been selected in a same branch of P, such that $\mathcal{R}_s$ occurs above $\mathcal{R}_{s+1}$ in the branch;

JJ) having **R**, a sequence $\mathcal{D}$ of ordered pairs of formula occurrences $\langle (t_1, t'_1), \ldots, (t_h, t'_h) \rangle$ is chosen, such that t_j occurs as subterm in the auxiliary proposition of $\mathcal{R}_j$ and t'_j occurs as subterm in the principal proposition of $\mathcal{R}_j$. t_1 is the $\mathcal{D}$ ***-axiom***, t_h is the $\mathcal{D}$ ***-theorem***.

The cardinality h of **R** is the ***length*** of $\mathcal{D}$.
If A_γ is the $\mathcal{D}$ *-axiom* and B_δ.is the $\mathcal{D}$ *-theorem* the ***inferential type*** **$\tau(\mathcal{D})$ of** $\mathcal{D}$ is $\gamma \rightarrow \delta$**.**

Abstract Deduction Structures (ADS) [Gentilini-Martelli IC Section 3]

A *choice criterion* for a Comp-ADS in P is a Δ_0–formula in the language of Primitive Recursive Arithmetic **PRA that expresses, for each element of the ADS, a set of recursive properties that it must have with respect to P**. A set of P-Comp ADS's admitting **PRA**-equivalent choice criteria constitutes *a kind of Comph- ADS's in P*.

By the *Comp ADS*'s we extend the notion of deduction in a way that emphasizes the Comp-rules action.

It is *not required that the elements of the ADS are* **LK$_\omega$**-*theorem*s in the standard sense, and their types may be arbitrary

Thus , we have at disposal two forms of deduction (inference): the standard *sequent deduction*, and the non-standard *abstract deduction*

Such duplicity allows to produce an inferential **but** referential semantics for **LK$_\omega$: the interpreting sequent trees belong to LK$_\omega$ -syntax but the semantical role is played (mainly) by the non-standard deduction, which is not included in LK$_\omega$**

Inferential models through Comp- ADS's (i.e. non-standard deduction)

Having defined the Comp-ADS's, an informal sketch of the inferential models can be stated:

formulas of full higher order logic are modeled through instances of the Comprehension Axiom, i.e. Comp-rule instances

by employing domains formed by ***LK***ω*- sequent trees, in which suitable kinds of Comp-ADS's occur, such that each tree is canonically associated to a closed* ***LK***ω*- formula.*

The most significant sequence of Comp-rules in the tree, i.e. the main Comp-ADS , represents the type and the logical structure of the modeled formula.

Moreover, a constructive ***inferential algebra*** *between the sequent trees of the domains is defined, so that logical connectives are modeled by corresponding operations between trees.*

Remark that we construct a referential/denotational semantics: the sequent trees in the inferential algebra (even if syntactically formed by two forms of deduction, the standard one and the abstract one) are external object w.r.t the interpreted language (the non-standard deduction is not defined in LKω syntax)

The ADS kind on which inferential semantics is based

The main challenge is to select a relevant and expressive kind of Comp ADS in a sequent tree.

Such problem is solved by introducing the ADS kind ***Kr*** of the ***critical chains***

[See Gentilini-Martelli IC Section 6]

$$\begin{array}{c l}
A_{o\to(o\to o)}B_oC_o \vdash A_{o\to(o\to o)}B_oC_o & \\ \hline
A_{o\to(o\to o)}B_oC_o \vdash \exists y_{o\to(o\to o)}[y_{o\to(o\to o)}B_oC_o] & \exists\text{-R} \\ \hline
A_{o\to(o\to o)}B_oC_o \vdash \exists z_{o\to o}\exists y_{o\to(o\to o)}[z_{o\to o}C_o] & \exists\text{-R} \\ \hline
\vdash A_{o\to(o\to o)}B_oC_o \supset \exists z_{o\to o}\exists y_{o\to(o\to o)}[z_{o\to o}C_o] & \supset\text{-R} \\ \hline
\vdash \exists w_o[A_{o\to(o\to o)}B_oC_o \supset w_o] & \exists\text{-R} \\ \hline
\neg\exists w_o[A_{o\to(o\to o)}B_oC_o \supset w_o] \vdash & \neg\text{-L} \\ \hline
\forall x_o\neg\exists w_o[x_o \supset w_o] \vdash & \forall\text{-L}
\end{array}$$

The following is a **critical chain** :

C: $<(A_{o\to(o\to o)},\ y_{o\to(o\to o)}),\ (y_{o\to(o\to o)}B_o,\ z_{o\to o}),\ (\exists z_{o\to o}\exists y_{o\to(o\to o)}[z_{o\to o}C_o],\ w_o),\ (A_{o\to(o\to o)}B_oC_o,\ x_o)>$

C-axiom: $A_{o\to(o\to o)}$; **C**-theorem: $A_{o\to(o\to o)}B_oC_o$; *length*(**C**)= 4;

inferential type(**C**)= $(o\to(o\to o))\to o$

Inferential Algebras Comp-ADS based [Gentilini-Martelli IC Section 3]

K a kind of Comp-ADS;
then the class of K-*based inferential domains* $\{{}^{K}D_{\alpha}\}_{\alpha\in types}$ of *modular* **LK$_{\omega}$**-trees, and the *inferential algebra*
$<g, \{{}^{K}D_{\alpha}\}_{\alpha\in types}, \Rightarrow, *>$ on $\cup_{\alpha\in types}\, {}^{K}D_{\alpha}$ are defined as follows:

0) A recursive bijection:
$g_{(\alpha,\beta)}:\{A_{\alpha} : A_{\alpha} \textit{ closed formula}\}\times\{A_{\beta} : A_{\beta} \textit{ closed formula}\}\rightarrow\{A_{\alpha\rightarrow\beta} : A_{\alpha\rightarrow\beta} \textit{ closed formula}\}$ is given, which is the *canonical bijection* of the algebra

i) ${}^{K}D_{o}$ (resp. ${}^{K}D_{i}$) is a denumerable set of **LK$_{\omega}$- sequent trees that have a main Comp-ADS of kind K, and have K-inferential type *o* (resp. *i*)**; this is the set of *K-modular trees of inferential type o* (resp. *i*).

ii) ${}^{K}D_{\gamma}$ set of K-modular trees of inferential type γ, ${}^{K}D_{\delta}$ set of K-modular trees of inferential type δ; then a **recursive procedure** $\Rightarrow$ is given, including the recursive bijection *g* as a component, **acting on the pairs $\langle Q_{\gamma}, Q_{\delta}\rangle \in {}^{K}D_{\gamma}\times{}^{K}D_{\delta}$, such that $\Rightarrow\langle Q_{\gamma}, Q_{\delta}\rangle$ is a tree of K-inferential type $\gamma\rightarrow\delta$**

Abstraction operation between trees:

The set of the trees of the form $\Rightarrow\langle Q_\gamma,Q_\delta\rangle$ gives the domain ${}^{K}\mathbf{D}_{\gamma\to\delta}$.

$\Rightarrow\langle Q_\gamma,Q_\delta\rangle$ is also written $Q_\gamma\Rightarrow Q_\delta$ **and $\Rightarrow$is called the *abstraction operation* of the inferential algebra;** Q_γ, Q_δ are the *abstracted trees* of the *abstraction tree* $Q_\gamma\Rightarrow Q_\delta$

Application operation between trees

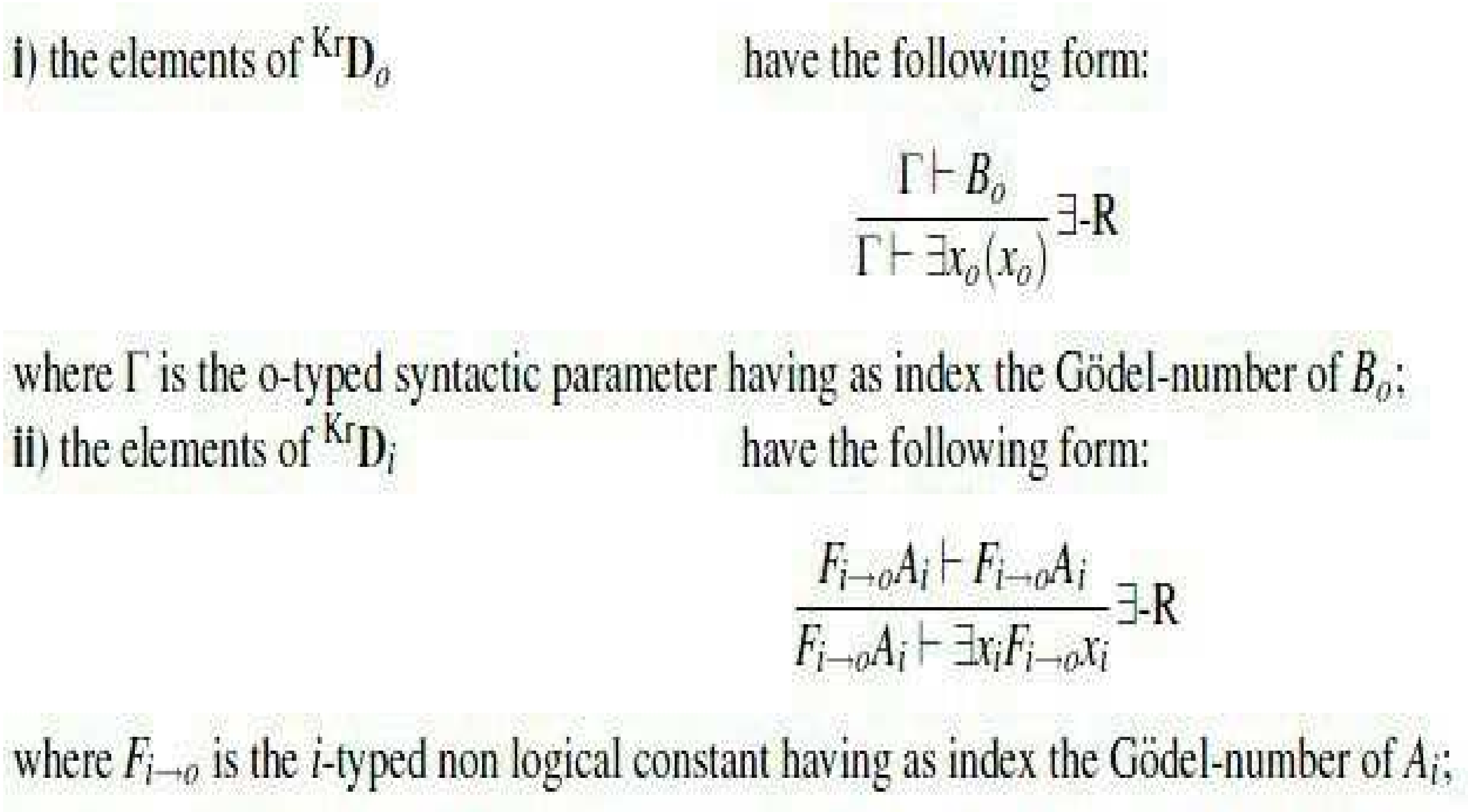
Having the domains $\{{}^{K}\mathrm{D}_\alpha\}_{\alpha\in types}$, for each pair of types γ and $\gamma\to\delta$ a recursive function

$$*:\ {}^{K}\mathrm{D}_{\gamma\to\delta}\times{}^{K}\mathrm{D}_{\gamma}\to{}^{K}\mathrm{D}_{\delta}$$

is given, such that the tree $*\langle Q_{\gamma\to\delta},Q_\gamma\rangle$ is a tree of K-inferential type δ, also written $Q_{\gamma\to\delta}*Q_\gamma$ **.**

$*$ is the *application operation*. $Q_{\gamma\to\delta}$, Q_γ are the *contracted trees* **.**

The set $\cup_{\alpha\in types}\ {}^{K}\mathrm{D}_\alpha$ with the operations abstraction $\Rightarrow$ and application $*$ is called an *inferential algebra of* **LKω-** *sequent trees based on the Comp-ADS’s of kind K*.



i) the elements of ${}^{\mathrm{Kr}}\mathbf{D}_o$ have the following form:

$$\frac{\Gamma \vdash B_o}{\Gamma \vdash \exists x_o(x_o)}\,\exists\text{-R}$$

where Γ is the o-typed syntactic parameter having as index the Gödel-number of B_o;

ii) the elements of ${}^{\mathrm{Kr}}\mathbf{D}_i$ have the following form:

$$\frac{F_{i\to o}A_i \vdash F_{i\to o}A_i}{F_{i\to o}A_i \vdash \exists x_i F_{i\to o}x_i}\,\exists\text{-R}$$

where $F_{i\to o}$ is the *i*-typed non logical constant having as index the Gödel-number of A_i;

Let $\mathbf{P}^{\mathbf{o}}$ and $\mathbf{P}^{\mathbf{i}}$ be the trees above

$\mathbf{P}^{i\to o}$, i.e. $\gamma \equiv i, \delta \equiv o$:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\{F^m_{i\to o}A_i \vdash F^m_{i\to o}A_i : m=1,\ldots,t\}\\ \Pi_1\\ X \vdash Y, F[\ldots\mathbf{A}_{\mathbf{i}}\ldots]\end{array}}{X \vdash Y, \exists x_i F[\ldots\mathbf{x}_{\mathbf{i}}\ldots]}\exists\text{-R} \qquad \dfrac{\begin{array}{c}\{\Gamma^r \vdash E_{i\to o}w_i : r=1,\ldots,k\}\\ \Pi_2\\ U \vdash V, N[\ldots E_{i\to o}w_i\ldots]\end{array}}{U \vdash V, N[\ldots E_{i\to o}w_i\ldots] \vee d_o}\vee\text{-R}}{X, U \vdash Y, V, \exists x_i F[\ldots x_i\ldots] \wedge (N[\ldots\mathbf{E}_{\mathbf{i}\to\mathbf{o}}\mathbf{w}_{\mathbf{i}}\ldots] \vee d_o)}\wedge\text{-R}}{X, U \vdash Y, V, \exists u_o(\exists x_i F[\ldots x_i\ldots] \wedge (N[\ldots\mathbf{u}_{\mathbf{o}}\ldots] \vee d_o))}\exists\text{-R}$$

The tree $\mathbf{P}^{\mathbf{i}\to\mathbf{o}}$ is the abstraction tree $\mathbf{P}^{\mathbf{i}} \Rightarrow \mathbf{P}^{\mathbf{o}}$

The tree $\mathrm{P}(\forall_{(o\to o)\to o}) \equiv \mathrm{P}(F_{o\to o}) \Rightarrow \mathrm{P}(G_o)$ is the following

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash C_o}{\Gamma \vdash C_o \vee h_o}\vee\text{-R}}{\Gamma \vdash \exists x_o(x_o \vee h_o)}\exists\text{-R} \qquad \dfrac{\Delta \vdash F_{o\to o}y_o}{\Delta \vdash F_{o\to o}y_o \vee b_o}\vee\text{-R}}{\Gamma,\Delta \vdash \exists x_o(x_o \vee h_o) \wedge (F_{o\to o}y_o \vee b_o)}\wedge\text{-R}}{\Gamma,\Delta \vdash \exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o))}\exists\text{-R}}{\Gamma,\Delta \vdash \exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (F_{o\to o}y_o \vee b_o))}\vee\text{-R}}{\Gamma,\Delta \vdash \exists q_{o\to o}(\exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (q_{o\to o} \vee b_o)))}\exists\text{-R}$$

(consider the root of the proof segment above as the left premise of the $\wedge$-R rule below)

$$\dfrac{\dfrac{\dfrac{\Omega \vdash \forall_{(o\to o)\to o}\theta_{o\to o}}{\Omega \vdash \forall_{(o\to o)\to o}\theta_{o\to o} \vee d_o}\vee\text{-R}}{\Gamma,\Delta,\Omega \vdash \exists q_{o\to o}(\exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (q_{o\to o}y_o \vee b_o))) \wedge (\forall_{(o\to o)\to o}\theta_{o\to o} \vee d_o)}\wedge\text{-R}}{\Gamma,\Delta,\Omega \vdash \exists r_o(\exists q_{o\to o}(\exists u_o(\exists x_o(x_o \vee h_o) \wedge (u_o \vee b_o)) \vee (\exists x_o(x_o \vee h_o) \wedge (q_{o\to o}y_o \vee b_o))) \wedge (r_o \vee d_o))}\exists\text{-R}$$

Inferential Interpretations of typed formulas in the domains of an Inferential Algebra

K a fixed suitable kind of Comp-ADS's

$<g, \{{}^{K}D_{\alpha}\}_{\alpha \in types}, \Rightarrow, *>$ inferential algebra, **then**

an *inferential interpretation for* $\mathbf{LK_{\omega}}$ *based on* K is an injective function:

$$V: \ \cup_{\alpha \in types}\{B_{\alpha} \mid B_{\alpha} \textit{ closed formula}\} \rightarrow <g, \{{}^{K}D_{\alpha}\}_{\alpha \in types}, \Rightarrow, *>$$

such that

i) $\mathbf{V(B_{\alpha})} \in {}^{K}D_{\alpha}$, and **is a sequent-tree of K-inferential type** α**, having a unique main K-Comph-ADS** $\mathcal{T}$

ii) $\mathbf{B_{\alpha}}$ **is a leftmost sub-term of the theorem the main Comp-ADS** $\mathcal{T}$ **of** $\mathbf{V(B_{\alpha})}$: such element has therefore the form $B_{\alpha}F_{\gamma}$, F_{γ} atom (non-logical constant)

V is *functionally sound*, that is:

j) $V(A_{\gamma\rightarrow\delta}) * V(B_{\gamma}) = V(A_{\gamma\rightarrow\delta}B_{\gamma})$

jj) $V(B_{\gamma}) \Rightarrow V(C_{\delta})$ is identical to $V(g(B_{\gamma}, C_{\delta}))$

Inferential Interpretations: remark

It must be remarked that $B\gamma$ **never** occurs as a standard **LKω**-theorem proved alongside the tree $V(B\gamma)$, and in particular $V(B_O)$ is **never** a standard proof of B_O !!!!!!

Inferential interpretations V in the inferential algebras $\langle g, \{{}^{Kr}\mathbf{D}_{\alpha}\}_{\alpha \in types}, \Rightarrow, * \rangle$ based on the ADS kind Kr of the critical chains : example

V(B_o) has the form :

$$\dfrac{\Gamma \vdash B_o}{\Gamma \vdash \exists x_o\,(x_o)}\ \exists\text{-}R$$

V($\neg_{o\to o}$) has the form

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash C_o}{\Gamma \vdash C_o \vee h_o}\ \vee\text{-}R}{\Gamma \vdash \exists x_o\,(x_o \vee h_o)}\ \exists\text{-}R \qquad \dfrac{\Delta \vdash \neg_{o\to o} y_o}{\Delta \vdash \neg_{o\to o} y_o \vee b_o}\ \vee\text{-}R}{\dfrac{\Gamma, \Delta \vdash \exists x_o\,(x_o \vee h_o) \wedge (\neg_{o\to o} y_o \vee b_o)}{\Gamma, \Delta \vdash \exists u_o\,(\exists x_o\,(x_o \vee h_o) \wedge (u_o \vee b_o))}\ \exists\text{-}R}\ \wedge\text{-}R$$

the main critical chain is $\mathcal{T} \equiv \langle (C_o, x_o), (\neg_{o\to o} y_o, u_o) \rangle$ having inferential type $o \to o$

Inferential interpretations V in the inferential algebras $\boldsymbol{<g, \{{}^{Kr}\mathbf{D}_{\alpha}\}_{\alpha \in types}, \Rightarrow, * >}$**:**

Example

$V(\neg_{o\rightarrow o})*V(B_o)$ results as identical to $V(\neg_{o\rightarrow o}B_o)$ which has the form:

$$\frac{\Gamma\ /\text{——}\ \ \neg_{o\rightarrow o}B_o}{\Gamma\ /\text{----}\ \ \exists x_o\ (x_o)}\qquad \exists\text{- R}$$

the main critical chain is $<(\neg_{o\rightarrow o}B_o\ , x_o)>$

Inferential Semantics: frames and truth

$<g, \{^K D_\alpha\}_{\alpha \in types}, \Rightarrow, *>$ inferential algebra

V: $\bigcup_{\alpha \in types}\{B_\alpha : B_\alpha \textit{ closed formula}\} \to <g, \{^K D_\alpha\}_{\alpha \in types}, \Rightarrow, *>$ inf. interpretation

Λ = consistent system extending **LKω**
such that the language $\mathcal{L}$(Λ) is an expansion of $\mathcal{L}$(**LKω**) at most through *a denumerable set of new primitive non logical constants for each type γ.*

Then,
an ***inferential semantics frame*** for **LK**$_\omega$ is a triple $\langle(g, \{^K D_\alpha\}, V), \Lambda, \cong\rangle$ where $\cong$ is the *semantical identity relation* established in the algebra *by the frame* , that is so defined:

$V(A_\alpha) \cong V(B_\alpha)$ iff a proof Q in Λ exists of the sentence $A_\alpha =_\alpha B_\alpha$

Inferential Semantics: truth

An inferential frame $\langle (g,\{{}^{K}D_{\alpha}\}, V), \Lambda, \cong \rangle$ is a *sound functional denotation for the logical constants* of **LK**$_{\omega}$ iff the following conditions hold:

$[V(\wedge_{o\to(o\to o)})*V(B_o)]*V(C_o) \cong V(\top)$ iff $V(B_o) \cong V(\top)$ and $V(C_o) \cong V(\top)$

$[V(\wedge_{o\to(o\to o)})*V(B_o)]*V(C_o) \cong V(\bot)$ otherwise

$V(\neg_{o\to o})*V(A_o) \cong V(\top)$ iff $V(A_o) \cong V(\bot)$

$V(\neg_{o\to o})*V(A_o) \cong V(\bot)$ otherwise

and so on for the other connectives

If the frame is a sound functional denotation, then a closed *o*-typed formula A is *true* in the inferential structure $\langle (g,\{{}^{K}D_{\alpha}\}, V), \Lambda, \cong \rangle$ iff $V(A) \cong V(\top)$, it is *false* iff $V(A) \cong V(\bot)$.

It must be remarked that Δ /—— $B =_{o} \top$ is a necessary but not a sufficient condition for the truth of *B*. The central requirements are the properties of the inferential algebra that allow $\mathcal{M}$ to be a sound denotation.

Founding results for inferential semantics

Main Theorem (existence of inferential algebras allowing inferential intepretations)

A kind K of Comp-ADS's is definable such that :

i) *The set of inferential algebras of modular* $\mathbf{LK}_\omega$*-trees* $<g, \{{}^{K}D_{\alpha}\}_{\alpha\in types}, \Rightarrow, *>$ *based on K is not empty*

ii) *An algebra* $\mathcal{A} \equiv <g, \{{}^{K}D_{\alpha}\}_{\alpha\in types}, \Rightarrow, *>$ *is definable, having all the properties required for an the image of an inferential interpretation, and such that the set of the functionally sound inferential interpretations is not empty*

Each logical operation between formulas $B\delta$, $A\gamma$, interpreted as follows

if $\blacksquare$ is a binary logical connective then $(\blacksquare B\delta)A\gamma$ is interpreted by $(V(\blacksquare)*V(B\delta))*V(A\gamma)$

if $\blacktriangle$ is a monadic logical connective then $\blacktriangle B\delta$ is interpreted by $V(\blacktriangle)*V(B\delta)$

Inferential Semantics Theorem

A suitable kind K of Comph-ADS's exists such that:

i) *The class of frames based on* $\{{}^{K}D_{\alpha}\}$ *that are sound functional denotation for the logical constants is not empty*. ii) $\mathbf{LK}_\omega$ *is sound and complete with respect to the class of inferential frames (structures) based on* $\{{}^{K}D_{\alpha}\}$

LKω-Inferential Algebras as semantical environment for standard first order logic: <u>searching for a semantics of first oder proofs</u>

A new step in the inferential semantics research program is to investigate Type Theory **LKω** and Inferential Algebras as semantical environment for
standard first order logic

with the main aim of producing **a semantics of standard first order proofs ,**
and possibly, to develop the perspective of giving
a semantics of proofs of standard finite order logics

Relevant preliminary steps for the main goal are:

First, the investigation of the *complexity* of interpretation trees of first order formulas in the **LKω** -Inferential Algebras $\mathit{Inf}\text{-}\mathcal{A}_g$

then to show that the interpretation trees of first order logic can give a **sub-algebra** of **LKω** -Inferential Algebras allowing a **completeness theorem** for first order logic [so that such sub-algebra is a *new semantics* for standard predicate logic]

First order logic inside LK$_\omega$: the system LK1

The language of **LK1** is so defined:
The *well formed expressions* of **LK1** are Church-terms with the following constraints

- variables are only of type i
- λ-abstractions are only over variables of type i and on formulas of type o, i.e. have only the form $\lambda x_i A_o$ with type $i \to o$
- Quantifiers occur only with type $(i \to o) \to o$ i.e. with the form $\exists_{(i \to o) \to o}$, $\forall_{(i \to o) \to o}$
- if τ is a type occurrence in any **LK1**-espression, no occurrences of the type o in τ precede any occurrence of the type i in τ
- non-logical constants of **LK1** are only of a type τ such that:
either in τ the type o does not occur, or the type o has at most one occurrence as *tail* of τ; τ has a *condensed writing* of the form: $i \to i \to i \to \ldots \to u$ where u is a primitive type, and if u coincides with o then τ is not primitive (thus o-typed constants are excluded)

Deduction apparatus of LK1:

It is identical to **LK**$_{\omega-}$ deduction apparatus **with the constraint that rules are restricted** to sequents of **LK1**-formulas

The **λ-rule** must be preserved since it is necessary inside the Church-syntax

Reading the standard first order syntax inside LK1-syntax

- Standard first order formulas are the **LK1** expressions (Church terms) of type o
- Standard terms of first order logic are the **LK1** expressions of type i
- Standard predicate letters are **LK1** expressions that are non logical constants of the form $A_{i \to .... \to i \to o}$
- Standard function letters are **LK1** expressions that are non logical constants of the form $f_{i \to .... \to i \to i}$
- Standard logical connectives are the **LK1** logical constants

Remark Any Church sub-term of an **LK1-**expression is an **LK1-**expression

Sending LK1 into LK$_\omega$ inferential algebras

We will explore the whole **LK$_\omega$** and the inferential algebras $\mathit{Inf}\text{-}\mathcal{A}_g = \{< g, \{^{Kr}D_\alpha\}_{\alpha \in types}, \Rightarrow, * >\}$ based on the Comp-ADS kind *Kr* of the *critical chains*, (g varying in the set of the canonical recursive bijections), as a semantical environment for first order logic.

Since **LK1**-expressions are a subset of **LK$_\omega$**-formulas, any *inferential interpretation for* **LK$_\omega$**:

$$V: \cup_{\alpha \in types}\{ B_\alpha : B_\alpha \text{ closed formula } \} \rightarrow < g, \{^{Kr}D_\alpha\}_{\alpha \in types}, \Rightarrow, * >$$

also acts on **LK1**-expressions.

We consider here only the class of Kr-inferential algebras $\mathit{Inf}\text{-}\mathcal{A}_g$ and the class of Kr-inferential interpretations V **that allow the completeness theorem for LK$_\omega$.**

Thus, for any **LK1**-expression F_α, $V(F_\alpha)$ is sequent-tree in $\mathit{Inf}\text{-}\mathcal{A}_g$, V functionally sound.

Searching LK1 completeness inside $\mathit{Inf}\text{-}\mathcal{A}_g$

LK$_\omega$ is sound and complete w.r.t. $\mathit{Inf}\text{-}\mathcal{A}_g$ with the interpretation V,
but obviously, the completeness for **LK**$_\omega$ is not the same as the completeness for **LK1**.

The first states that if F_O is true in the $\mathit{Inf}\text{-}\mathcal{A}_g$ based frames then F_O is **LK**$_\omega$**-**provable,
the second would establish that F_O is LK1 provable too, which is a very different matter.

Semantical spectrum of a LK1- expression and complexity parameters

Definition Let $Inf\text{-}\mathcal{A}_g$ be any Kr-inferential algebra and V any inferential interpretation for $\mathbf{LK}_\omega$.
Let F_α be any **LK1**-expression. Then the **semantical spectrum** Spec(F_α) of F_α in $Inf\text{-}\mathcal{A}_g$ is the set of sequent trees {V(F_α), V(E_γ): E_γ is a Church sub-term of F_α}

In the trees V(B_α) of $Inf\text{-}\mathcal{A}_g$ we will consider the ADS's which are *critical chains* (both *strong* and *weak*) and the following *complexity parameters*:
- the *inferential types* of the critical chain ADS's
- the *length of the maximal critical chain* M(V(B_α)) of V(B_α) (which is the number of Comp rule occurrences involved by the chain)
- *the size of the tree* V(B_α) which is the highest number of proof lines in a branch

Complexity of inferential interpretations of LK1 logical and a-logical expressions

Definition

We call *logical expression of* **LK1** any Church term of **LK1** where at least one logical constant occurs

We call *a-logical expression of* **LK1** any Church term of **LK1** where no logical constants occur

Proposition *Let* F_α *be any* **LK1** *logical expression. Then*:

$$1 \leq \text{Size}(\text{V}(F_\alpha)) \leq 12$$

Proposition *Let* F_α *be any* **LK1** *logical expression. Then*:

$$1 \leq \text{length}\ [\text{M}(\text{V}(F_\alpha))] \leq 7$$

Thus, inferential interpretation trees of an LK1 logical expression are bounded both as standard deduction objects (size of the sequent tree) and as non-standard deduction objects (length of the maximal critical chain ADS)

Complexity of inferential interpretations of LK1 a-logical expressions

Proposition

Let C_γ *be any a-logical* **LK1**-*expression. Then, in general, the size of the trees in* Spec(C_γ) *is not bounded*

Proposition

Let C_γ be *any a-logical* **LK1**-*expression*. *Then*:

$$1 \leq \text{length}\,[\mathrm{M}(\mathrm{V}(C_\gamma))] \leq 3$$

Such results reflect some facts:

- the trees of Spec(C_γ) in $\mathit{Inf}\text{-}\mathcal{A}_{\mathrm{g}}$ also take into account the *aritiy* of any predicate letter or function letter occurring in C_γ
- the size of V(C_γ) depends on the *height* of the type γ, the length of M(V(C_γ)) depends on the *order* of the type γ

Moreover, it is remarkable that the maximum of the ADS semantical complexity of **LK1** logical expressions is greater than that of the **LK1** a-logical expressions

$\mathit{Inf}\text{-}\mathcal{A}_{\mathbf{g}}$ sub-algebras characterizing LK1 and completeness

Theorem

Let SPEC1 ≡ $\cup\{$ Spec($E\gamma$): $E\gamma$is an **LK1**-expression$\}$. Then a canonical bijection *h* exists such that SPEC1 is a sub-algebra of $\mathit{Inf}\text{-}\mathcal{A}_{\mathbf{h}}$ that allows the completeness of **LK1** w.r.t. the inferential semantics frames based on it.

We say that SPEC1, with the complexity bounds established by the stated Propositions, *characterizes first order logic*

APPENDIX

The abstract deduction structure *critical chain*

It is the **critical chain** ADS that allows to define suitable inferential interpretations for the soundness and completeness results

In order to introduce the critical chains, it is **necessary to define two kinds of ancestor-descendant relation among terms/formulas in a proof tree:**

(The details are complicated and only some essential points can be given)

i)An ancestor-descendant relation which links formulas of type *o* that also are isolated formulas: this is the *isolated ancestor-descendant relation*,
(*isolated a.d. relation*),
such relation has a strong role in the expression of the action of propositional logical rules, and mutually connects the auxiliary and principal propositions of a rule occurrence.

ii) An ancestor-descendant relation *between arbitrary terms*: it links terms possibly having different types: this is the *term ancestor-descendant relation* (*term a.d. relation*),
which has a strong role in the expression of the action of the Comp-rules and of the λ-rule. In this relation the type *o* has not any particular status

Strong and weak propositional inference

In order to define the notion of critical chain, a distinction between propositional inferences has to be done:

<u>the purpose is to include in a critical chain only the propositional logical inference which is the farthest from weakening rules,</u> i.e. having the uppermost ancestors of its auxiliary propositions introduced by axioms and not by weakenings.

Definition

We call *strong propositional rules* the rules $\vee$-L,

$\wedge$-R, $\supset$-L, $\neg$-R, $\neg$-L

and

weak propositional rules the rules $\vee$-R, $\wedge$-L, $\supset$-R.

ii) In an isolated ancestor-descendant chain in a proof P,

B is *a strong isolated descendant of A* if no element between *A* and *B* in the chain (possibly except *A*) is the principal proposition of a weak propositional rule,

B is a weak isolated descendant of A otherwise

Critical Chain ADS: definition (1)

The ADS kind Kr of the critical chains gives the inferential algebras $<g, \{{}^{Kr}D_{\alpha}\}_{\alpha \in types}, \Rightarrow, *>$ on which the Main Theorem and the Inferential Semantics Theorem are based

Let P be a **LK_{ω}-** proof tree. **A *critical chain $\mathcal{T}$ in P* is the Comp-ADS** so defined:

i) $\mathcal{T}$ can be written as a sequence of pairs of formulas occurring in P, where the types may range in the set of all types: $(A^1, B^1),\ldots,(A^m, B^m)$

ii) For each (A^j, B^j), **A^j is a maximal auxiliary term of a Comp-rule occurence** in P, and B^j is the principal term which is the successor of A^j

iii) (A^1, B^1) is called *initial pair of the chain*, A^1 is the axiom of the chain and has the following properties:

iiia) no term ancestor of A^1 occurs either in a strong isolated descendant of the principal proposition F of a Comp-rule occurrence in P, or in F itself

iiib) no term ancestor of A^1 is a principal term of a Comp-rule occurrence in P

Critical Chain ADS : definition (2)

iv) If (A^j, B^j), (A^{j+1}, B^{j+1}) are consecutive pairs in $\mathcal{T}$, one among the following conditions holds:

iva) B^j is a term ancestor of a sub-term of A^{j+1} through a term ancestor-descendant chain in which no element different from B^j is a principal term of a Comp-rule occurrence

ivb) B^j occurs as sub-term in A^{j+1}

ivc) A^{j+1} occurs either in a strong isolated descendant of the principal proposition F of the Comp-rule occurrence in which B^j is a principal term, or in F itself

v) The pair (A^m, B^m) *is the end-pair of* $\mathcal{T}$, where **A^m is the theorem of the chain** ; B^m is such that **it has not any term descendant which is sub-term of a maximal auxiliary term of a Comp-rule**. Since B^m is a bound variable occurrence, it has an integral term descendant either in the root of P, or in a cut formula.

Remark: In the definition above **the crucial point is iv)**, and it must be noted that: in iva), ivb) the link between Bj and Aj^{+1} is standardly given by a term a.d. chain, **while in ivc) the link may be provided by an isolated a.d. chain including strong propositional inference**

Critical chain ADS : comments

What is the essence of a critical chain? It is an ancestor-descendant relation among formulas of arbitrary types in a branch,

which witnesses the action of a sequence of Comp-rules, with possible substantial intrusions of the principal proposition of a strong propositional rule.

That is, propositional inference is mixed with the Comp-rule action, under the condition that it is completely extraneous with respect to the weakening.

Without this mix, a chain constituted only by a a term a.d. chain connecting auxiliary and principal terms of a sequence of Comp-rules distributed in a branch, would be much less expressive, and maybe trivial:
it would not express the real impact of a Comp-rule occurrence, i.e. the formulas really included in its action field.